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Michael Renardy
Robert C. Rogers

TEXTS IN APPLIED MATHEMATICS

**An Introduction to
Partial Differential
Equations**

Second Edition



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