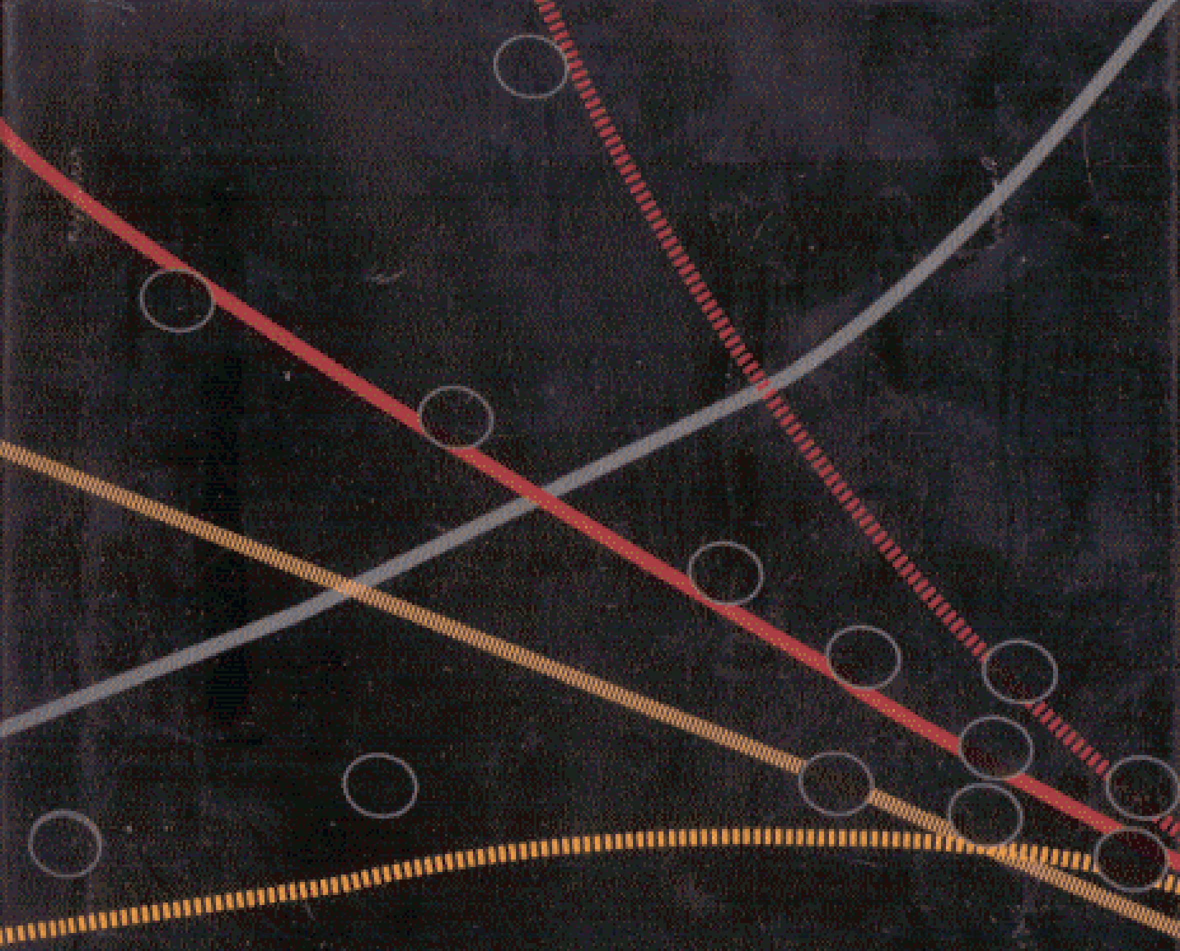

PATH INTEGRALS

in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets

Hagen KLEINERT

Contents

Preface vii

Preface to the Second Edition ix

Preface to the First Edition xi

1 Fundamentals 1

1.1 Classical Mechanics . . . 1
1.2 Relativistic Mechanics in Curved Spacetime . . . 11
1.3 Quantum Mechanics . . . 12
1.4 Dirac's Bra-Ket Formalism . . . 18
1.5 Observables . . . 26
1.6 Quantum Mechanics of General Lagrangian Systems . . . 31
1.7 Particle on the Surface of a Sphere . . . 37
1.8 Spinning Top . . . 39
1.9 Time Evolution Operator . . . 46
1.10 Properties of Time Evolution Operator . . . 49
1.11 Scattering . . . 56
1.11.1 Scattering Matrix . . . 56
1.11.2 Cross Section . . . 57
1.11.3 Born Approximation . . . 59
1.11.4 Partial Wave Expansion and Eikonal Approximation . . . 59
1.11.5 Scattering Amplitude from Time Evolution Amplitude . . . 61
1.11.6 Lippmann-Schwinger Equation . . . 62
1.12 Heisenberg Picture of Quantum Mechanics . . . 65
1.13 Classical and Quantum Statistics . . . 68
1.13.1 Canonical Ensemble . . . 69
1.13.2 Grand-Canonical Ensemble . . . 70
Appendix 1A The Asymmetric Top . . . 74
Notes and References . . . 76

2 Path Integrals — Elementary Properties and Simple Solutions 77

2.1 Path Integral Representation of Time Evolution Amplitudes . . . 77
2.2 Exact Solution for Free Particle . . . 87
2.3 Finite Slicing Properties of Free-Particle Amplitude . . . 96
2.4 Exact Solution for Harmonic Oscillator . . . 97

2.5 Useful Fluctuation Formulas . . . 102
2.6 Oscillator Amplitude on Finite Time Lattice . . . 104
2.7 Gelfand-Yaglom Formula . . . 106
2.7.1 Recursive Calculation of Fluctuation Determinant . . . 106
2.7.2 Examples . . . 107
2.7.3 Calculation on Unsliced Time Axis . . . 109
2.7.4 D'Alembert's Construction . . . 110
2.7.5 Another Simple Formula . . . 111
2.7.6 Generalization to D Dimensions . . . 113
2.8 Path Integral for Harmonic Oscillator with Arbitrary Time-Dependent Frequency . . . 114
2.8.1 Coordinate Space . . . 114
2.8.2 Momentum Space . . . 117
2.9 Free-Particle and Oscillator Wave Functions . . . 119
2.10 Path Integrals and Quantum Statistics . . . 121
2.11 Density Matrix . . . 123
2.12 Quantum Statistics of Harmonic Oscillator . . . 128
2.13 Time-Dependent Harmonic Potential . . . 134
2.14 Functional Measure in Fourier Space . . . 138
2.15 Classical Limit . . . 141
2.16 Calculation Techniques on Sliced Time Axis. Poisson Formula . . . 142
2.17 Field-Theoretic Definition of Harmonic Path Integral by Analytic Regularization . . . 147
2.17.1 Zero-Temperature Evaluation of Frequency Sum . . . 148
2.17.2 Finite-Temperature Evaluation of Frequency Sum . . . 151
2.17.3 Duality Transformation and Low-Temperature Expansion . . . 153
2.18 Finite-N Behavior of Thermodynamic Quantities . . . 159
2.19 Time Evolution Amplitude of Freely Falling Particle . . . 161
2.20 Charged Particle in Magnetic Field . . . 163
2.21 Charged Particle in Magnetic Field and Harmonic Potential . . . 168
2.22 Gauge Invariance and Alternative Path Integral Representation . . . 171
2.23 Velocity Path Integral . . . 172
2.24 Path Integral Representation of Scattering Matrix . . . 174
2.24.1 General Development . . . 174
2.24.2 Improved Formulation . . . 177
2.24.3 Eikonal Approximation to Scattering Amplitude . . . 178
Appendix 2A Derivation of Baker-Campbell-Hausdorff and Magnus Formulas . . . 179
Appendix 2B Direct Calculation of Time-Sliced Oscillator Amplitude . . . 182
Appendix 2C Derivation of Mehler Formula . . . 184
Notes and References . . . 184

3 External Sources, Correlations, and Perturbation Theory 187

3.1 External Sources . . . 187
3.2 Green Function of Harmonic Oscillator . . . 190
3.2.1 Wronski Construction . . . 191
3.2.2 Spectral Representation . . . 195
3.3 Green Functions of First-Order Differential Equation . . . 196
3.3.1 Time-Independent Frequency . . . 197
3.3.2 Time-Dependent Frequency . . . 204
3.4 Summing Spectral Representation of Green Function . . . 205
3.5 Wronski Construction for Periodic and Antiperiodic Green Functions 207
3.6 Time Evolution Amplitude in Presence of Source Term . . . 208
3.7 External Source in Quantum-Statistical Path Integral . . . 213
3.7.1 Continuation of Real-Time Result . . . 213
3.7.2 Calculation at Imaginary Time . . . 217
3.8 Lattice Green Function . . . 224
3.9 Correlation Functions, Generating Functional, and Wick Expansion 225
3.10 Correlation Functions of Charged Particle in Magnetic Field . . . 228
3.11 Correlation Functions in Canonical Path Integral . . . 229
3.11.1 Harmonic Correlation Functions . . . 230
3.11.2 Relations between Various Amplitudes . . . 233
3.11.3 Harmonic Generating Functionals . . . 234
3.12 Particle in Heat Bath . . . 237
3.13 Particle in Heat Bath of Photons . . . 241
3.14 Harmonic Oscillator in Heat Bath . . . 243
3.15 Perturbation Expansion of Anharmonic Systems . . . 246
3.16 Calculation of Perturbation Series with Feynman Diagrams . . . 249
3.17 Field-Theoretic Definition of Anharmonic Path Integral . . . 253
3.18 Generating Functional of Connected Correlation Functions . . . 254
3.18.1 Connectedness Structure of Correlation Functions . . . 255
3.18.2 Decomposition of Correlation Functions into Connected Correlation Functions . . . 258
3.18.3 Functional Generation of Vacuum Diagrams . . . 260
3.18.4 Correlation Functions from Vacuum Diagrams . . . 264
3.18.5 Generating Functional for Vertex Functions. Effective Action 266
3.18.6 Ginzburg-Landau Approximation to Generating Functional 271
3.18.7 Composite Fields . . . 272
3.19 Path Integral Calculation of Effective Action by Loop Expansion . 273
3.19.1 General Formalism . . . 273
3.19.2 Quadratic Fluctuations . . . 278
3.19.3 Effective Action to Second Order in $\hbar$ . . . 281
3.19.4 Background Field Method for Effective Action . . . 284
3.20 Nambu-Goldstone Theorem . . . 287
3.21 Effective Classical Potential . . . 289
3.21.1 Effective Classical Boltzmann Factor . . . 291

3.21.2 High- and Low-Temperature Behavior 293
3.21.3 Alternative Candidate for Effective Classical Potential . . . 295
3.21.4 Harmonic Correlation Function without Zero Mode 296
3.21.5 Perturbation Expansion 296
3.21.6 First-Order Perturbative Result 298
3.22 Perturbative Calculation of Scattering Amplitude 300
3.22.1 Generating Functional 300
3.22.2 Application to Scattering Amplitude 301
3.22.3 First Correction to Eikonal Approximation 302
3.23 Rayleigh-Schrödinger Perturbation Expansion 303
3.23.1 Energy Levels 303
3.23.2 Scattering Amplitudes 308
3.24 Functional Determinants from Green Functions 309
Appendix 3A Feynman Integrals for $T \neq 0$ 315
Appendix 3B Energy Shifts for $gx^4/4$-Interaction 318
Appendix 3C Matrix Elements for General Potential 320
Appendix 3D Level-Shifts from Schrödinger Equation 322
Appendix 3E Recursion Relations for Perturbation Coefficients 324
3E.1 One-Dimensional Interaction x^4 324
3E.2 Interaction r^4 in D-Dimensional Radial Oscillator 328
3E.3 Interaction r^{2q} in D Dimensions 329
3E.4 Polynomial Interaction in D Dimensions 329
Notes and References 329

4 Semiclassical Time Evolution Amplitude 332
4.1 The Wentzel-Kramers-Brillouin (WKB) Approximation 332
4.2 Saddle Point Approximation 336
4.2.1 Ordinary Integrals 337
4.2.2 Path Integrals 339
4.3 Van Vleck-Pauli-Morette Determinant 345
4.4 Fundamental Composition Law for Semiclassical Time Evolution Amplitude 349
4.5 Semiclassical Fixed-Energy Amplitude 351
4.6 Semiclassical Amplitude in Momentum Space 353
4.7 Semiclassical Quantum-Mechanical Partition Function 355
4.8 Multi-Dimensional Systems 360
4.9 Quantum Corrections to Classical Density of States 365
4.10 Thomas-Fermi Model of Neutral Atoms 370
4.10.1 Semiclassical Limit 370
4.10.2 Quantum Correction Near Origin 378
4.10.3 Exchange Energy 380
4.10.4 Higher Quantum Corrections to Thomas-Fermi Energies . 382
4.11 Classical Action of Coulomb System 387
4.12 Semiclassical Scattering 396

4.12.1 General Formulation 396
4.12.2 Semiclassical Cross Section of Mott Scattering 399
Notes and References 400

5 Variational Perturbation Theory **403**
5.1 Variational Approach to Effective Classical Partition Function . . . 403
5.2 Local Harmonic Trial Partition Function 404
5.3 Optimal Upper Bound 409
5.4 Accuracy of Variational Approximation 410
5.5 Weakly Bound Ground State Energy in Finite-Range Potential Well 412
5.6 Possible Direct Generalizations 414
5.7 Effective Classical Potential for Anharmonic Oscillator 415
5.8 Particle Densities 421
5.9 Extension to D Dimensions 424
5.10 Application to Coulomb and Yukawa Potentials 426
5.11 Hydrogen Atom in Strong Magnetic Field 429
5.11.1 Weak-Field Behavior 433
5.11.2 Effective Classical Potential 433
5.12 Effective Potential and Magnetization Curves 436
5.13 Variational Approach to Excitation Energies 438
5.14 Systematic Improvement of Feynman-Kleinert Approximation . . . 443
5.15 Applications of Variational Perturbation Expansion 446
5.15.1 Anharmonic Oscillator at $T = 0$ 446
5.15.2 Anharmonic Oscillator for $T > 0$ 448
5.16 Convergence of Variational Perturbation Expansion 452
5.17 Variational Perturbation Theory for Strong-Coupling Expansion . . 459
5.18 General Strong-Coupling Expansions 462
5.19 Variational Interpolation between Weak and Strong-Coupling Expansions 465
5.20 Systematic Improvement of Excited Energies 467
5.21 Variational Treatment of Double-Well Potential 468
5.22 Higher-Order Effective Classical Potential for Nonpolynomial Interactions 470
5.22.1 Evaluation of Path Integrals 471
5.22.2 Higher-Order Smearing Formula in D Dimensions 472
5.22.3 Isotropic Second-Order Approximation to Coulomb Problem 474
5.22.4 Anisotropic Second-Order Approximation to Coulomb Problem 476
5.22.5 Zero-Temperature Limit 478
5.23 Polarons 482
5.23.1 Partition Function 484
5.23.2 Harmonic Trial System 486
5.23.3 Effective Mass 492
5.23.4 Second-Order Correction 492

5.23.5 Polaron in Magnetic Field, Bipolarons, etc. 493
5.23.6 Variational Interpolation for Polaron Energy and Mass . . 494
5.24 Density Matrices 497
5.24.1 Harmonic Oscillator 498
5.24.2 Variational Perturbation Theory for Density Matrices . . . 499
5.24.3 Smearing Formula for Density Matrices 501
5.24.4 First-Order Variational Results 503
5.24.5 Smearing Formula in Higher Spatial Dimensions 508
5.24.6 Applications 510
Appendix 5A Feynman Integrals for $T \neq 0$ without Zero Frequency . . . 521
Appendix 5B Proof of Scaling Relation for the Extrema of W_N 523
Appendix 5C Second-Order Shift of Polaron Energy 526
Notes and References 527

6 Path Integrals with Topological Constraints 532
6.1 Point Particle on Circle 532
6.2 Infinite Wall 536
6.3 Point Particle in Box 541
6.4 Strong-Coupling Theory for Particle in Box 543
6.4.1 Partition Function 544
6.4.2 Perturbation Expansion 544
6.4.3 Variational Strong-Coupling Approximations 547
6.4.4 Special Properties of Expansion 549
6.4.5 Exponentially Fast Convergence 550
Notes and References 551

7 Many Particle Orbits — Statistics and Second Quantization 553
7.1 Ensembles of Bose and Fermi Particle Orbits 554
7.2 Bose-Einstein Condensation 561
7.2.1 Free Bose Gas 561
7.2.2 Effect of Interactions 570
7.2.3 Bose-Einstein Condensation in Harmonic Trap 575
7.2.4 Interactions in Harmonic Trap 584
7.3 Gas of Free Fermions 588
7.4 Statistics Interaction 593
7.5 Fractional Statistics 598
7.6 Second-Quantized Bose Fields 599
7.7 Fluctuating Bose Fields 602
7.8 Coherent States 608
7.9 Dimensional Regularization of Functional Determinants 612
7.10 Second-Quantized Fermi Fields 615
7.11 Fluctuating Fermi Fields 616
7.11.1 Grassmann Variables 616
7.11.2 Fermionic Functional Determinant 619

7.11.3 Coherent States for Fermions 623
7.12 Hilbert Space of Quantized Grassmann Variable 624
7.12.1 Single Real Grassmann Variable 625
7.12.2 Quantizing Harmonic Oscillator with Grassmann Variables 628
7.12.3 Spin System with Grassmann Variables 629
7.13 External Sources in a^*, a -Path Integral 633
7.14 Generalization to Pair Terms 635
7.15 Spatial Degrees of Freedom 637
7.15.1 Grand-Canonical Ensemble of Particle Orbits from Free Fluctuating Field 637
7.15.2 First versus Second Quantization 638
7.15.3 Interacting Fields 639
7.15.4 Effective Classical Field Theory 640
Notes and References 642

8 Path Integrals in Spherical Coordinates **646**
8.1 Angular Decomposition in Two Dimensions 646
8.2 Trouble with Feynman's Path Integral Formula in Radial Coordinates 649
8.3 Cautionary Remarks 653
8.4 Time Slicing Corrections 656
8.5 Angular Decomposition in Three and More Dimensions 661
8.5.1 Three Dimensions 661
8.5.2 D Dimensions 664
8.6 Radial Path Integral for Harmonic Oscillator and Free Particle 669
8.7 Particle near the Surface of a Sphere in D Dimensions 670
8.8 Angular Barriers near the Surface of a Sphere 673
8.8.1 Angular Barriers in Three Dimensions 673
8.8.2 Angular Barriers in Four Dimensions 678
8.9 Motion on a Sphere in D Dimensions 683
8.10 Path Integrals on Group Spaces 687
8.11 Path Integral of a Spinning Top 690
Notes and References 691

9 Fixed-Energy Amplitude and Wave Functions **693**
9.1 General Relations 693
9.2 Free Particle in D Dimensions 696
9.3 Harmonic Oscillator in D Dimensions 699
9.4 Free Particle from $\omega \to 0$ -Limit of Oscillator 705
9.5 Charged Particle in Uniform Magnetic Field 707
Notes and References 714

10 Spaces with Curvature and Torsion **715**
10.1 Einstein's Equivalence Principle 716
10.2 Classical Motion of Mass Point in General Metric-Affine Space . . 717

10.2.1 Equations of Motion 717
10.2.2 Nonholonomic Mapping to Spaces with Torsion 720
10.2.3 New Equivalence Principle 725
10.2.4 Classical Action Principle for Spaces with Curvature and Torsion 726
10.3 Path Integral in Metric-Affine Space 730
10.3.1 Nonholonomic Transformation of Action 731
10.3.2 Measure of Path Integration 735
10.4 Completing Solution of Path Integral on Surface of Sphere 741
10.5 External Potentials and Vector Potentials 742
10.6 Perturbative Calculation of Path Integrals in Curved Space 745
10.6.1 Free and Interacting Parts of Action 745
10.6.2 Zero Temperature 747
10.7 Model Study of Coordinate Invariance 750
10.7.1 Diagrammatic Expansion 751
10.7.2 Diagrammatic Expansion in d Time Dimensions 753
10.8 Calculating Loop Diagrams 754
10.8.1 Reformulation in Configuration Space 761
10.8.2 Integrals over Products of Two Distributions 762
10.8.3 Integrals over Products of Four Distributions 763
10.9 Distributions as Limits of Bessel Function 766
10.9.1 Correlation Function and Derivatives 766
10.9.2 Integrals over Products of Two Distributions 767
10.9.3 Integrals over Products of Four Distributions 769
10.10 Simple Rules for Calculating Singular Integrals 771
10.11 Perturbative Calculation on Finite Time Intervals 776
10.11.1 Diagrammatic Elements 777
10.11.2 Cumulant Expansion of D-Dimensional Free-Particle Amplitude in Curvilinear Coordinates 778
10.11.3 Propagator in $1-\varepsilon$ Time Dimensions 780
10.11.4 Coordinate Independence for Dirichlet Boundary Conditions 782
10.11.5 Time Evolution Amplitude in Curved Space 789
10.11.6 Covariant Results for Arbitrary Coordinates 795
10.12 Effective Classical Potential in Curved Space 802
10.12.1 Covariant Fluctuation Expansion 803
10.12.2 Arbitrariness of q_0^μ 805
10.12.3 Zero-Mode Properties 807
10.12.4 Covariant Perturbation Expansion 810
10.12.5 Covariant Result from Noncovariant Expansion 811
10.12.6 Particle on Unit Sphere 814
10.13 Covariant Effective Action for Quantum Particle with Coordinate-Dependent Mass 817
10.13.1 Formulating the Problem 817
10.13.2 Derivative Expansion 820

Appendix 10A Nonholonomic Gauge Transformations in Electromagnetism 823
10A.1 Gradient Representation of Magnetic Field of Current Loop 824
10A.2 Generating Magnetic Fields by Multivalued Gauge Transformations . . . 827
10A.3 Magnetic Monopoles . . . 828
10A.4 Minimal Magnetic Coupling of Particles from Multivalued Gauge Transformations . . . 829
10A.5 Gauge Field Representation of Current Loops . . . 831
Appendix 10B Difference between Multivalued Basis Tetrads and Vierbein Fields . . . 833
Appendix 10C Cancellation of Powers of $\delta(0)$ . . . 835
Notes and References . . . 838

11 Schrödinger Equation in General Metric-Affine Spaces 843
11.1 Integral Equation for Time Evolution Amplitude . . . 843
11.1.1 From the Recursion Relation to Schrödinger's Equation . . 844
11.1.2 Alternative Evaluation . . . 847
11.2 Equivalent Path Integral Representations . . . 850
11.3 Potentials and Vector Potentials . . . 854
11.4 Unitarity Problem . . . 855
11.5 Alternative Attempts . . . 857
11.6 DeWitt-Seeley Expansion of Time Evolution Amplitude . . . 858
Appendix 11A Cancellations in Effective Potential . . . 862
Appendix 11B DeWitt's Amplitude . . . 865
Notes and References . . . 865

12 New Path Integral Formula for Singular Potentials 867
12.1 Path Collapse in Feynman's formula for the Coulomb System . . . 867
12.2 Stable Path Integral with Singular Potentials . . . 870
12.3 Time-Dependent Regularization . . . 875
12.4 Relation with Schrödinger Theory. Wave Functions . . . 877
Notes and References . . . 879

13 Path Integral of Coulomb System 880
13.1 Pseudotime Evolution Amplitude . . . 880
13.2 Solution for the Two-Dimensional Coulomb System . . . 882
13.3 Absence of Time Slicing Corrections for $D = 2$ . . . 887
13.4 Solution for the Three-Dimensional Coulomb System . . . 893
13.5 Absence of Time Slicing Corrections for $D = 3$ . . . 899
13.6 Geometric Argument for Absence of Time Slicing Corrections . . . 903
13.7 Comparison with Schrödinger Theory . . . 904
13.8 Angular Decomposition of Amplitude, and Radial Wave Functions 909
13.9 Remarks on Geometry of Four-Dimensional u^{μ}-Space . . . 913
13.10 Solution in Momentum Space . . . 915

13.10.1 Gauge-Invariant Canonical Path Integral 915
13.10.2 Another Form of Action 918
13.10.3 Absence of Extra R-Term 919
Appendix 13A Group-Theoretic Aspects of Coulomb States 919
Notes and References 924

14 Solution of Further Path Integrals by the Duru-Kleinert Method 925
14.1 One-Dimensional Systems 925
14.2 Derivation of the Effective Potential 928
14.3 Comparison with Schrödinger Quantum Mechanics 933
14.4 Applications 933
14.4.1 Radial Harmonic Oscillator and Morse System 934
14.4.2 Radial Coulomb System and Morse System 936
14.4.3 Equivalence of Radial Coulomb System and Radial Oscillator 937
14.4.4 Angular Barrier near Sphere, and Rosen-Morse Potential . 945
14.4.5 Angular Barrier near Four-Dimensional Sphere, and General Rosen-Morse Potential 948
14.4.6 Hulthén Potential and General Rosen-Morse Potential . . . 951
14.4.7 Extended Hulthén Potential and General Rosen-Morse Potential 953
14.5 D-Dimensional Systems 954
14.6 Path Integral of the Dionium Atom 955
14.6.1 Formal Solution 956
14.6.2 Absence of Time Slicing Corrections 959
14.7 Time-Dependent Duru-Kleinert Transformation 963
Appendix 14A Affine Connection of Dionium Atom 967
Appendix 14B Algebraic Aspects of Dionium States 967
Notes and References 968

15 Path Integrals in Polymer Physics 969
15.1 Polymers and Ideal Random Chains 969
15.2 Moments of End-to-End Distribution 971
15.3 Exact End-to-End Distribution in Three Dimensions 974
15.4 Short-Distance Expansion for a Long Polymer 976
15.5 Saddle Point Approximation to Three-Dimensional End-to-End Distribution 978
15.6 Path Integral for Continuous Gaussian Distribution 979
15.7 Stiff Polymers 981
15.7.1 Path Integral 983
15.7.2 Moments of End-to-End Distribution 984
15.8 Schrödinger Equation and Recursive Solution for Moments 988
15.8.1 Recursive Solution of Schrödinger Equation. 989
15.8.2 Approximation to End-to-End Distribution 992
15.8.3 From Moments to End-to-End Distribution for $D = 3$ . . . 997

15.9 Excluded-Volume Effects 999
15.10 Flory's Argument 1006
15.11 Polymer Field Theory 1007
15.12 Fermi Fields for Self-Avoiding Lines 1015
Notes and References 1015

16 Polymers and Particle Orbits in Multiply Connected Spaces 1018
16.1 Simple Model for Entangled Polymers 1018
16.2 Entangled Fluctuating Particle Orbit: Aharonov-Bohm Effect . . . 1021
16.3 Aharonov-Bohm Effect and Fractional Statistics 1032
16.4 Self-Entanglement of Polymer 1037
16.5 The Gauss Invariant of Two Curves 1051
16.6 Bound States of Polymers — Ribbons 1054
16.7 Chern-Simons Theory of Entanglements 1060
16.8 Entangled Pair of Polymers 1063
16.8.1 Polymer Field Theory for Probabilities 1066
16.8.2 Calculation of Partition Function 1068
16.8.3 Calculation of Numerator in Second Moment 1070
16.8.4 First Diagram in Fig. 16.23 1071
16.8.5 Second and Third Diagrams in Fig. 16.23 1073
16.8.6 Fourth Diagram in Fig. 16.23 1074
16.8.7 Second Topological Moment 1075
16.9 Chern-Simons Theory of Statistical Interaction 1076
16.10 Second-Quantized Anyon Fields 1078
16.11 Fractional Quantum Hall Effect 1082
16.12 Anyonic Superconductivity 1085
16.13 Non-Abelian Chern-Simons Theory 1087
Appendix 16A Calculation of Feynman Diagrams for Polymer Entanglement 1090
Appendix 16B Kauffman and BLM/Ho polynomials 1091
Appendix 16C Skein Relation between Wilson Loop Integrals 1092
Appendix 16D London Equations 1095
Appendix 16E Hall Effect in Electron Gas 1097
Notes and References 1097

17 Tunneling 1103
17.1 Double-Well Potential 1103
17.2 Classical Solutions — Kinks and Antikinks 1106
17.3 Quadratic Fluctuations 1110
17.3.1 Zero-Eigenvalue Mode 1116
17.3.2 Continuum Part of Fluctuation Factor 1119
17.4 General Formula for Eigenvalue Ratios 1122
17.5 Fluctuation Determinant from Classical Solution 1124
17.6 Wave Functions of Double-Well 1127
17.7 Gas of Kinks and Antikinks and Level Splitting Formula 1128

17.8 Fluctuation Correction to Level Splitting . . . 1133
17.9 Tunneling and Decay . . . 1138
17.10 Large-Order Behavior of Perturbation Expansions . . . 1147
17.10.1 Growth Properties of Expansion Coefficients . . . 1148
17.10.2 Semiclassical Large-Order Behavior . . . 1151
17.10.3 Fluctuation Correction to the Imaginary Part and Large-Order Behavior . . . 1156
17.10.4 Variational Approach to Tunneling. Perturbation Coefficients to All Orders . . . 1159
17.10.5 Convergence of Variational Perturbation Expansion . . . 1167
17.11 Decay of Supercurrent in Thin Closed Wire . . . 1176
17.12 Decay of Metastable Thermodynamic Phases . . . 1187
17.13 Decay of Metastable Vacuum State in Quantum Field Theory . . . 1194
17.14 Crossover from Quantum Tunneling to Thermally Driven Decay . . . 1196
Appendix 17A Feynman Integrals for Fluctuation Correction . . . 1197
Notes and References . . . 1200

18 Nonequilibrium Quantum Statistics **1203**
18.1 Linear Response and Time-Dependent Green Functions for $T \neq 0$. 1203
18.2 Spectral Representations of $T \neq 0$ Green Functions . . . 1206
18.3 Other Important Green Functions . . . 1209
18.4 Hermitian Adjoint Operators . . . 1212
18.5 Harmonic Oscillator Green Functions for $T \neq 0$ . . . 1213
18.5.1 Creation Annihilation Operators . . . 1213
18.5.2 Real Field Operators . . . 1216
18.6 Nonequilibrium Green Functions . . . 1218
18.7 Perturbation Theory for Nonequilibrium Green Functions . . . 1227
18.8 Path Integral Coupled to Thermal Reservoir . . . 1230
18.9 Fokker-Planck Equation . . . 1235
18.9.1 Canonical Path Integral for Probability Distribution . . . 1236
18.9.2 Solving the Operator Ordering Problem . . . 1238
18.9.3 Strong Damping . . . 1244
18.10 Langevin Equations . . . 1247
18.11 Stochastic Calculus . . . 1250
18.12 Supersymmetry . . . 1256
18.13 Stochastic Quantum Liouville Equation . . . 1258
18.14 Relation to Quantum Langevin Equation . . . 1261
18.15 Electromagnetic Dissipation and Decoherence . . . 1261
18.15.1 Forward–Backward Path Integral . . . 1262
18.16 Master Equation for Time Evolution . . . 1266
18.17 Line Width . . . 1269
18.18 Lamb shift . . . 1270
18.19 Langevin Equations . . . 1274
18.20 Fokker-Planck Equation in Spaces with Curvature and Torsion . . . 1275

18.21 Stochastic Interpretation of Quantum-Mechanical Amplitudes . . . 1277
18.22 Stochastic Equation for Schrödinger Wave Function 1279
18.23 Real Stochastic and Deterministic Equation for Schrödinger Wave Function 1281
18.23.1 Stochastic Differential Equation 1281
18.23.2 Equation for Noise Average 1282
18.23.3 Harmonic Oscillator 1283
18.23.4 General Potential 1283
18.23.5 Deterministic Equation 1284
18.24 Heisenberg Picture for Probability Evolution 1285
Appendix 18A Inequalities for Diagonal Green Functions 1289
Appendix 18B General Generating Functional 1293
Appendix 18C Wick Decomposition of Operator Products 1298
Notes and References 1300

19 Relativistic Particle Orbits **1303**
19.1 Special Features of Relativistic Path Integrals 1305
19.2 Proper Action for Fluctuating Relativistic Particle Orbits 1308
19.2.1 Gauge-Invariant Formulation 1308
19.2.2 Simplest Gauge Fixing 1310
19.2.3 Partition Function of Ensemble of Closed Particle Loops . 1311
19.2.4 Fixed-Energy Amplitude 1313
19.3 Relativistic Coulomb System 1313
19.4 Relativistic Particle in Electromagnetic Field 1317
19.4.1 Action and Partition Function 1317
19.4.2 Perturbation Expansion 1318
19.4.3 Lowest-Order Vacuum Polarization 1320
19.5 Path Integral for Spin-1/2 Particle 1324
19.5.1 Dirac Theory 1324
19.5.2 Path Integral 1328
19.5.3 Amplitude with Electromagnetic Interaction 1330
19.5.4 Effective Action in Electromagnetic Field 1333
19.5.5 Perturbation Expansion 1334
19.5.6 Vacuum Polarization 1335
19.6 Supersymmetry 1337
19.6.1 Global Invariance 1337
19.6.2 Local Invariance 1338
Notes and References 1340

20 Path Integrals and Financial Markets **1342**
20.1 Fluctuation Properties of Financial Assets 1342
20.1.1 Harmonic Approximation to Fluctuations 1344
20.1.2 Lévy Distributions 1346
20.1.3 Truncated Lévy Distributions 1347

20.1.4 Asymmetric Truncated Lévy Distributions 1352
20.1.5 Meixner Distributions 1355
20.1.6 Other Non-Gaussian Distributions 1356
20.1.7 Lévy-Khintchine Formula 1360
20.1.8 Debye-Waller Factor for Non-Gaussian Fluctuations 1361
20.1.9 Path Integral for Non-Gaussian Distribution 1361
20.1.10 Fokker-Planck-Type Equation 1363
20.2 Martingales 1368
20.2.1 Gaussian Martingales 1368
20.2.2 Non-Gaussian Martingales 1369
20.3 Origin of Heavy Tails 1371
20.3.1 Pair of Stochastic Differential Equations 1371
20.3.2 Fokker-Planck Equation 1372
20.3.3 Solution of Fokker-Planck Equation 1375
20.3.4 Pure x-Distribution 1376
20.3.5 Long-Time Behavior 1378
20.3.6 Tail Behavior for all Times 1382
20.3.7 Path Integral Calculation 1384
20.3.8 Natural Martingales 1385
20.4 Option Pricing 1386
20.4.1 Black-Scholes Option Pricing Model 1387
20.4.2 Evolution Equations of Portfolios with Options 1389
20.4.3 Option Pricing for Gaussian Fluctuations 1391
20.4.4 Option Pricing for Non-Gaussian Fluctuations 1395
20.4.5 Option Pricing for Fluctuating Variance 1398
20.4.6 Perturbation Expansion and Smile 1400
Appendix 20A Large-x Behavior of Truncated Lévy Distribution 1403
Appendix 20B Gaussian Weight 1407
Appendix 20C Comparison with Dow-Jones Data 1408
Notes and References 1409

Index **1417**