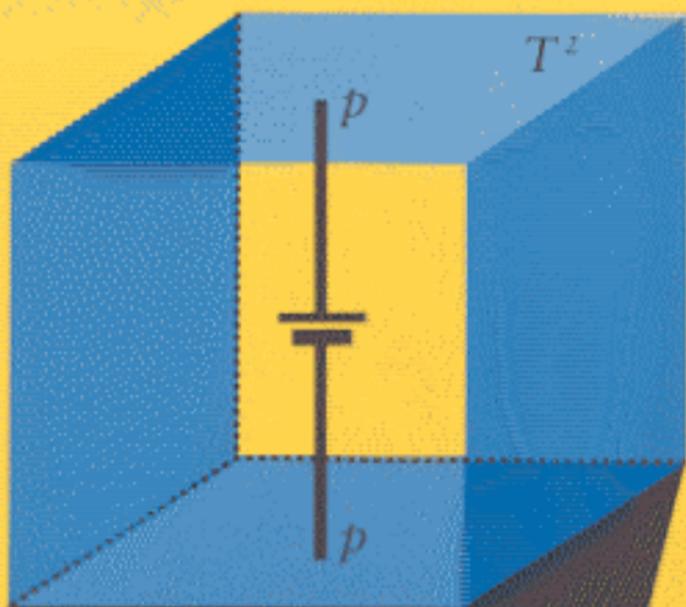




THE GEOMETRY OF PHYSICS

AN INTRODUCTION
SECOND EDITION



THEODORE FRANKEL

Contents

<i>Preface to the Second Edition</i>	page xix
<i>Preface to the Revised Printing</i>	xxi
<i>Preface to the First Edition</i>	xxiii

I Manifolds, Tensors, and Exterior Forms

1 Manifolds and Vector Fields	3
1.1. Submanifolds of Euclidean Space	3
1.1a. Submanifolds of \mathbb{R}^N	4
1.1b. The Geometry of Jacobian Matrices: The "Differential"	7
1.1c. The Main Theorem on Submanifolds of \mathbb{R}^N	8
1.1d. A Nontrivial Example: The Configuration Space of a Rigid Body	9
1.2. Manifolds	11
1.2a. Some Notions from Point Set Topology	11
1.2b. The Idea of a Manifold	13
1.2c. A Rigorous Definition of a Manifold	19
1.2d. Complex Manifolds: The Riemann Sphere	21
1.3. Tangent Vectors and Mappings	22
1.3a. Tangent or "Contravariant" Vectors	23
1.3b. Vectors as Differential Operators	24
1.3c. The Tangent Space to M^n at a Point	25
1.3d. Mappings and Submanifolds of Manifolds	26
1.3e. Change of Coordinates	29
1.4. Vector Fields and Flows	30
1.4a. Vector Fields and Flows on \mathbb{R}^n	30
1.4b. Vector Fields on Manifolds	33
1.4c. Straightening Flows	34

2	Tensors and Exterior Forms	37
2.1.	Covectors and Riemannian Metrics	37
2.1a.	Linear Functionals and the Dual Space	37
2.1b.	The Differential of a Function	40
2.1c.	Scalar Products in Linear Algebra	42
2.1d.	Riemannian Manifolds and the Gradient Vector	45
2.1e.	Curves of Steepest Ascent	46
2.2.	The Tangent Bundle	48
2.2a.	The Tangent Bundle	48
2.2b.	The Unit Tangent Bundle	50
2.3.	The Cotangent Bundle and Phase Space	52
2.3a.	The Cotangent Bundle	52
2.3b.	The Pull-Back of a Covector	52
2.3c.	The Phase Space in Mechanics	54
2.3d.	The Poincaré 1-Form	56
2.4.	Tensors	58
2.4a.	Covariant Tensors	58
2.4b.	Contravariant Tensors	59
2.4c.	Mixed Tensors	60
2.4d.	Transformation Properties of Tensors	62
2.4e.	Tensor Fields on Manifolds	63
2.5.	The Grassmann or Exterior Algebra	66
2.5a.	The Tensor Product of Covariant Tensors	66
2.5b.	The Grassmann or Exterior Algebra	66
2.5c.	The Geometric Meaning of Forms in \mathbb{R}^n	70
2.5d.	Special Cases of the Exterior Product	70
2.5e.	Computations and Vector Analysis	71
2.6.	Exterior Differentiation	73
2.6a.	The Exterior Differential	73
2.6b.	Examples in \mathbb{R}^3	75
2.6c.	A Coordinate Expression for d	76
2.7.	Pull-Backs	77
2.7a.	The Pull-Back of a Covariant Tensor	77
2.7b.	The Pull-Back in Elasticity	80
2.8.	Orientation and Pseudoforms	82
2.8a.	Orientation of a Vector Space	82
2.8b.	Orientation of a Manifold	83
2.8c.	Orientability and 2-Sided Hypersurfaces	84
2.8d.	Projective Spaces	85
2.8e.	Pseudoforms and the Volume Form	85
2.8f.	The Volume Form in a Riemannian Manifold	87
2.9.	Interior Products and Vector Analysis	89
2.9a.	Interior Products and Contractions	89
2.9b.	Interior Product in \mathbb{R}^3	90
2.9c.	Vector Analysis in \mathbb{R}^3	92

2.10.	Dictionary	94
3	Integration of Differential Forms	95
3.1.	Integration over a Parameterized Subset	95
3.1a.	Integration of a p -Form in \mathbb{R}^p	95
3.1b.	Integration over Parameterized Subsets	96
3.1c.	Line Integrals	97
3.1d.	Surface Integrals	99
3.1e.	Independence of Parameterization	101
3.1f.	Integrals and Pull-Backs	102
3.1g.	Concluding Remarks	102
3.2.	Integration over Manifolds with Boundary	104
3.2a.	Manifolds with Boundary	105
3.2b.	Partitions of Unity	106
3.2c.	Integration over a Compact Oriented Submanifold	108
3.2d.	Partitions and Riemannian Metrics	109
3.3.	Stokes's Theorem	110
3.3a.	Orienting the Boundary	110
3.3b.	Stokes's Theorem	111
3.4.	Integration of Pseudoforms	114
3.4a.	Integrating Pseudo- n -Forms on an n -Manifold	115
3.4b.	Submanifolds with Transverse Orientation	115
3.4c.	Integration over a Submanifold with Transverse Orientation	116
3.4d.	Stokes's Theorem for Pseudoforms	117
3.5.	Maxwell's Equations	118
3.5a.	Charge and Current in Classical Electromagnetism	118
3.5b.	The Electric and Magnetic Fields	119
3.5c.	Maxwell's Equations	120
3.5d.	Forms and Pseudoforms	122
4	The Lie Derivative	125
4.1.	The Lie Derivative of a Vector Field	125
4.1a.	The Lie Bracket	125
4.1b.	Jacobi's Variational Equation	127
4.1c.	The Flow Generated by $[X, Y]$	129
4.2.	The Lie Derivative of a Form	132
4.2a.	Lie Derivatives of Forms	132
4.2b.	Formulas Involving the Lie Derivative	134
4.2c.	Vector Analysis Again	136
4.3.	Differentiation of Integrals	138
4.3a.	The Autonomous (Time-Independent) Case	138
4.3b.	Time-Dependent Fields	140
4.3c.	Differentiating Integrals	142
4.4.	A Problem Set on Hamiltonian Mechanics	145
4.4a.	Time-Independent Hamiltonians	147

4.4b.	Time-Dependent Hamiltonians and Hamilton's Principle	151
4.4c.	Poisson Brackets	154
5	The Poincaré Lemma and Potentials	155
5.1.	A More General Stokes's Theorem	155
5.2.	Closed Forms and Exact Forms	156
5.3.	Complex Analysis	158
5.4.	The Converse to the Poincaré Lemma	160
5.5.	Finding Potentials	162
6	Holonomic and Nonholonomic Constraints	165
6.1.	The Frobenius Integrability Condition	165
6.1a.	Planes in \mathbb{R}^3	165
6.1b.	Distributions and Vector Fields	167
6.1c.	Distributions and 1-Forms	167
6.1d.	The Frobenius Theorem	169
6.2.	Integrability and Constraints	172
6.2a.	Foliations and Maximal Leaves	172
6.2b.	Systems of Mayer-Lie	174
6.2c.	Holonomic and Nonholonomic Constraints	175
6.3.	Heuristic Thermodynamics via Caratheodory	178
6.3a.	Introduction	178
6.3b.	The First Law of Thermodynamics	179
6.3c.	Some Elementary Changes of State	180
6.3d.	The Second Law of Thermodynamics	181
6.3e.	Entropy	183
6.3f.	Increasing Entropy	185
6.3g.	Chow's Theorem on Accessibility	187
II Geometry and Topology		
7	\mathbb{R}^3 and Minkowski Space	191
7.1.	Curvature and Special Relativity.	191
7.1a.	Curvature of a Space Curve in \mathbb{R}^3	191
7.1b.	Minkowski Space and Special Relativity	192
7.1c.	Hamiltonian Formulation	196
7.2.	Electromagnetism in Minkowski Space	196
7.2a.	Minkowski's Electromagnetic Field Tensor	196
7.2b.	Maxwell's Equations	198
8	The Geometry of Surfaces in \mathbb{R}^3	201
8.1.	The First and Second Fundamental Forms	201
8.1a.	The First Fundamental Form, or Metric Tensor	201
8.1b.	The Second Fundamental Form	203
8.2.	Gaussian and Mean Curvatures	205
8.2a.	Symmetry and Self-Adjointness	205

8.2b.	Principal Normal Curvatures	206
8.2c.	Gauss and Mean Curvatures: The Gauss Normal Map	207
8.3.	The Brouwer Degree of a Map: A Problem Set	210
8.3a.	The Brouwer Degree	210
8.3b.	Complex Analytic (Holomorphic) Maps	214
8.3c.	The Gauss Normal Map Revisited: The Gauss–Bonnet Theorem	215
8.3d.	The Kronecker Index of a Vector Field	215
8.3e.	The Gauss Looping Integral	218
8.4.	Area, Mean Curvature, and Soap Bubbles	221
8.4a.	The First Variation of Area	221
8.4b.	Soap Bubbles and Minimal Surfaces	226
8.5.	Gauss's <i>Theorema Egregium</i>	228
8.5a.	The Equations of Gauss and Codazzi	228
8.5b.	The <i>Theorema Egregium</i>	230
8.6.	Geodesics	232
8.6a.	The First Variation of Arc Length	232
8.6b.	The Intrinsic Derivative and the Geodesic Equation	234
8.7.	The Parallel Displacement of Levi-Civita	236
9	Covariant Differentiation and Curvature	241
9.1.	Covariant Differentiation	241
9.1a.	Covariant Derivative	241
9.1b.	Curvature of an Affine Connection	244
9.1c.	Torsion and Symmetry	245
9.2.	The Riemannian Connection	246
9.3.	Cartan's Exterior Covariant Differential	247
9.3a.	Vector-Valued Forms	247
9.3b.	The Covariant Differential of a Vector Field	248
9.3c.	Cartan's Structural Equations	249
9.3d.	The Exterior Covariant Differential of a Vector-Valued Form	250
9.3e.	The Curvature 2-Forms	251
9.4.	Change of Basis and Gauge Transformations	253
9.4a.	Symmetric Connections Only	253
9.4b.	Change of Frame	253
9.5.	The Curvature Forms in a Riemannian Manifold	255
9.5a.	The Riemannian Connection	255
9.5b.	Riemannian Surfaces M^2	257
9.5c.	An Example	257
9.6.	Parallel Displacement and Curvature on a Surface	259
9.7.	Riemann's Theorem and the Horizontal Distribution	263
9.7a.	Flat Metrics	263
9.7b.	The Horizontal Distribution of an Affine Connection	263
9.7c.	Riemann's Theorem	266

10 Geodesics	269
10.1. Geodesics and Jacobi Fields	269
10.1a. Vector Fields Along a Surface in M^n	269
10.1b. Geodesics	271
10.1c. Jacobi Fields	272
10.1d. Energy	274
10.2. Variational Principles in Mechanics	275
10.2a. Hamilton's Principle in the Tangent Bundle	275
10.2b. Hamilton's Principle in Phase Space	277
10.2c. Jacobi's Principle of "Least" Action	278
10.2d. Closed Geodesics and Periodic Motions	281
10.3. Geodesics, Spiders, and the Universe	284
10.3a. Gaussian Coordinates	284
10.3b. Normal Coordinates on a Surface	287
10.3c. Spiders and the Universe	288
11 Relativity, Tensors, and Curvature	291
11.1. Heuristics of Einstein's Theory	291
11.1a. The Metric Potentials	291
11.1b. Einstein's Field Equations	293
11.1c. Remarks on Static Metrics	296
11.2. Tensor Analysis	298
11.2a. Covariant Differentiation of Tensors	298
11.2b. Riemannian Connections and the Bianchi Identities	299
11.2c. Second Covariant Derivatives: The Ricci Identities	301
11.3. Hilbert's Action Principle	303
11.3a. Geodesics in a Pseudo-Riemannian Manifold	303
11.3b. Normal Coordinates, the Divergence and Laplacian	303
11.3c. Hilbert's Variational Approach to General Relativity	305
11.4. The Second Fundamental Form in the Riemannian Case	309
11.4a. The Induced Connection and the Second Fundamental Form	309
11.4b. The Equations of Gauss and Codazzi	311
11.4c. The Interpretation of the Sectional Curvature	313
11.4d. Fixed Points of Isometries	314
11.5. The Geometry of Einstein's Equations	315
11.5a. The Einstein Tensor in a (Pseudo-)Riemannian Space-Time	315
11.5b. The Relativistic Meaning of Gauss's Equation	316
11.5c. The Second Fundamental Form of a Spatial Slice	318
11.5d. The Codazzi Equations	319
11.5e. Some Remarks on the Schwarzschild Solution	320
12 Curvature and Topology: Synge's Theorem	323
12.1. Synge's Formula for Second Variation	324
12.1a. The Second Variation of Arc Length	324
12.1b. Jacobi Fields	326

12.2.	Curvature and Simple Connectivity	329
12.2a.	Syngé's Theorem	329
12.2b.	Orientability Revisited	331
13	Betti Numbers and De Rham's Theorem	333
13.1.	Singular Chains and Their Boundaries	333
13.1a.	Singular Chains	333
13.1b.	Some 2-Dimensional Examples	338
13.2.	The Singular Homology Groups	342
13.2a.	Coefficient Fields	342
13.2b.	Finite Simplicial Complexes	343
13.2c.	Cycles, Boundaries, Homology, and Betti Numbers	344
13.3.	Homology Groups of Familiar Manifolds	347
13.3a.	Some Computational Tools	347
13.3b.	Familiar Examples	350
13.4.	De Rham's Theorem	355
13.4a.	The Statement of De Rham's Theorem	355
13.4b.	Two Examples	357
14	Harmonic Forms	361
14.1.	The Hodge Operators	361
14.1a.	The $*$ Operator	361
14.1b.	The Codifferential Operator $\delta = d^*$	364
14.1c.	Maxwell's Equations in Curved Space-Time M^4	366
14.1d.	The Hilbert Lagrangian	367
14.2.	Harmonic Forms	368
14.2a.	The Laplace Operator on Forms	368
14.2b.	The Laplacian of a 1-Form	369
14.2c.	Harmonic Forms on Closed Manifolds	370
14.2d.	Harmonic Forms and De Rham's Theorem	372
14.2e.	Bochner's Theorem	374
14.3.	Boundary Values, Relative Homology, and Morse Theory	375
14.3a.	Tangential and Normal Differential Forms	376
14.3b.	Hodge's Theorem for Tangential Forms	377
14.3c.	Relative Homology Groups	379
14.3d.	Hodge's Theorem for Normal Forms	381
14.3e.	Morse's Theory of Critical Points	382

III Lie Groups, Bundles, and Chern Forms

15	Lie Groups	391
15.1.	Lie Groups, Invariant Vector Fields, and Forms	391
15.1a.	Lie Groups	391
15.1b.	Invariant Vector Fields and Forms	395
15.2.	One-Parameter Subgroups	398
15.3.	The Lie Algebra of a Lie Group	402
15.3a.	The Lie Algebra	402

15.3b.	The Exponential Map	403
15.3c.	Examples of Lie Algebras	404
15.3d.	Do the 1-Parameter Subgroups Cover G ?	405
15.4.	Subgroups and Subalgebras	407
15.4a.	Left Invariant Fields Generate Right Translations	407
15.4b.	Commutators of Matrices	408
15.4c.	Right Invariant Fields	409
15.4d.	Subgroups and Subalgebras	410
16	Vector Bundles in Geometry and Physics	413
16.1.	Vector Bundles	413
16.1a.	Motivation by Two Examples	413
16.1b.	Vector Bundles	415
16.1c.	Local Trivializations	417
16.1d.	The Normal Bundle to a Submanifold	419
16.2.	Poincaré's Theorem and the Euler Characteristic	421
16.2a.	Poincaré's Theorem	422
16.2b.	The Stiefel Vector Field and Euler's Theorem	426
16.3.	Connections in a Vector Bundle	428
16.3a.	Connection in a Vector Bundle	428
16.3b.	Complex Vector Spaces	431
16.3c.	The Structure Group of a Bundle	433
16.3d.	Complex Line Bundles	433
16.4.	The Electromagnetic Connection	435
16.4a.	Lagrange's Equations without Electromagnetism	435
16.4b.	The Modified Lagrangian and Hamiltonian	436
16.4c.	Schrödinger's Equation in an Electromagnetic Field	439
16.4d.	Global Potentials	443
16.4e.	The Dirac Monopole	444
16.4f.	The Aharonov–Bohm Effect	446
17	Fiber Bundles, Gauss–Bonnet, and Topological Quantization	451
17.1.	Fiber Bundles and Principal Bundles	451
17.1a.	Fiber Bundles	451
17.1b.	Principal Bundles and Frame Bundles	453
17.1c.	Action of the Structure Group on a Principal Bundle	454
17.2.	Coset Spaces	456
17.2a.	Cosets	456
17.2b.	Grassmann Manifolds	459
17.3.	Chern's Proof of the Gauss–Bonnet–Poincaré Theorem	460
17.3a.	A Connection in the Frame Bundle of a Surface	460
17.3b.	The Gauss–Bonnet–Poincaré Theorem	462
17.3c.	Gauss–Bonnet as an Index Theorem	465
17.4.	Line Bundles, Topological Quantization, and Berry Phase	465
17.4a.	A Generalization of Gauss–Bonnet	465
17.4b.	Berry Phase	468
17.4c.	Monopoles and the Hopf Bundle	473

18	Connections and Associated Bundles	475
18.1.	Forms with Values in a Lie Algebra	475
18.1a.	The Maurer–Cartan Form	475
18.1b.	\mathfrak{g} -Valued p -Forms on a Manifold	477
18.1c.	Connections in a Principal Bundle	479
18.2.	Associated Bundles and Connections	481
18.2a.	Associated Bundles	481
18.2b.	Connections in Associated Bundles	483
18.2c.	The Associated Ad Bundle	485
18.3.	r -Form Sections of a Vector Bundle: Curvature	488
18.3a.	r -Form Sections of E	488
18.3b.	Curvature and the Ad Bundle	489
19	The Dirac Equation	491
19.1.	The Groups $SO(3)$ and $SU(2)$	491
19.1a.	The Rotation Group $SO(3)$ of \mathbb{R}^3	492
19.1b.	$SU(2)$: The Lie Algebra $\mathfrak{su}(2)$	493
19.1c.	$SU(2)$ Is Topologically the 3-Sphere	495
19.1d.	$Ad : SU(2) \rightarrow SO(3)$ in More Detail	496
19.2.	Hamilton, Clifford, and Dirac	497
19.2a.	Spinors and Rotations of \mathbb{R}^3	497
19.2b.	Hamilton on Composing Two Rotations	499
19.2c.	Clifford Algebras	500
19.2d.	The Dirac Program: The Square Root of the d' Alembertian	502
19.3.	The Dirac Algebra	504
19.3a.	The Lorentz Group	504
19.3b.	The Dirac Algebra	509
19.4.	The Dirac Operator $\not{\partial}$ in Minkowski Space	511
19.4a.	Dirac Spinors	511
19.4b.	The Dirac Operator	513
19.5.	The Dirac Operator in Curved Space–Time	515
19.5a.	The Spinor Bundle	515
19.5b.	The Spin Connection in \mathfrak{SM}	518
20	Yang–Mills Fields	523
20.1.	Noether's Theorem for Internal Symmetries	523
20.1a.	The Tensorial Nature of Lagrange's Equations	523
20.1b.	Boundary Conditions	526
20.1c.	Noether's Theorem for Internal Symmetries	527
20.1d.	Noether's Principle	528
20.2.	Weyl's Gauge Invariance Revisited	531
20.2a.	The Dirac Lagrangian	531
20.2b.	Weyl's Gauge Invariance Revisited	533
20.2c.	The Electromagnetic Lagrangian	534
20.2d.	Quantization of the A Field: Photons	536

20.3.	The Yang–Mills Nucleon	537
20.3a.	The Heisenberg Nucleon	537
20.3b.	The Yang–Mills Nucleon	538
20.3c.	A Remark on Terminology	540
20.4.	Compact Groups and Yang–Mills Action	541
20.4a.	The Unitary Group Is Compact	541
20.4b.	Averaging over a Compact Group	541
20.4c.	Compact Matrix Groups Are Subgroups of Unitary Groups	542
20.4d.	<i>Ad</i> Invariant Scalar Products in the Lie Algebra of a Compact Group	543
20.4e.	The Yang–Mills Action	544
20.5.	The Yang–Mills Equation	545
20.5a.	The Exterior Covariant Divergence ∇^*	545
20.5b.	The Yang–Mills Analogy with Electromagnetism	547
20.5c.	Further Remarks on the Yang–Mills Equations	548
20.6.	Yang–Mills Instantons	550
20.6a.	Instantons	550
20.6b.	Chern's Proof Revisited	553
20.6c.	Instantons and the Vacuum	557
21	Betti Numbers and Covering Spaces	561
21.1.	Bi-invariant Forms on Compact Groups	561
21.1a.	Bi-invariant p -Forms	561
21.1b.	The Cartan p -Forms	562
21.1c.	Bi-invariant Riemannian Metrics	563
21.1d.	Harmonic Forms in the Bi-invariant Metric	564
21.1e.	Weyl and Cartan on the Betti Numbers of G	565
21.2.	The Fundamental Group and Covering Spaces	567
21.2a.	Poincaré's Fundamental Group $\pi_1(M)$	567
21.2b.	The Concept of a Covering Space	569
21.2c.	The Universal Covering	570
21.2d.	The Orientable Covering	573
21.2e.	Lifting Paths	574
21.2f.	Subgroups of $\pi_1(M)$	575
21.2g.	The Universal Covering Group	575
21.3.	The Theorem of S. B. Myers: A Problem Set	576
21.4.	The Geometry of a Lie Group	580
21.4a.	The Connection of a Bi-invariant Metric	580
21.4b.	The Flat Connections	581
22	Chern Forms and Homotopy Groups	583
22.1.	Chern Forms and Winding Numbers	583
22.1a.	The Yang–Mills "Winding Number"	583
22.1b.	Winding Number in Terms of Field Strength	585
22.1c.	The Chern Forms for a $U(n)$ Bundle	587

22.2.	Homotopies and Extensions	591
22.2a.	Homotopy	591
22.2b.	Covering Homotopy	592
22.2c.	Some Topology of $SU(n)$	594
22.3.	The Higher Homotopy Groups $\pi_k(M)$	596
22.3a.	$\pi_k(M)$	596
22.3b.	Homotopy Groups of Spheres	597
22.3c.	Exact Sequences of Groups	598
22.3d.	The Homotopy Sequence of a Bundle	600
22.3e.	The Relation between Homotopy and Homology Groups	603
22.4.	Some Computations of Homotopy Groups	605
22.4a.	Lifting Spheres from M into the Bundle P	605
22.4b.	$SU(n)$ Again	606
22.4c.	The Hopf Map and Fibring	606
22.5.	Chern Forms as Obstructions	608
22.5a.	The Chern Forms c_r for an $SU(n)$ Bundle Revisited	608
22.5b.	c_2 as an "Obstruction Cocycle"	609
22.5c.	The Meaning of the Integer $j(\Delta_4)$	612
22.5d.	Chern's Integral	612
22.5e.	Concluding Remarks	615
Appendix A. Forms in Continuum Mechanics		617
A.a.	The Classical Cauchy Stress Tensor and Equations of Motion	617
A.b.	Stresses in Terms of Exterior Forms	618
A.c.	Symmetry of Cauchy's Stress Tensor in \mathbb{R}^n	620
A.d.	The Piola–Kirchhoff Stress Tensors	622
A.e.	Stored Energy of Deformation	623
A.f.	Hamilton's Principle in Elasticity	626
A.g.	Some Typical Computations Using Forms	629
A.h.	Concluding Remarks	635
Appendix B. Harmonic Chains and Kirchhoff's Circuit Laws		636
B.a.	Chain Complexes	636
B.b.	Cochains and Cohomology	638
B.c.	Transpose and Adjoint	639
B.d.	Laplacians and Harmonic Cochains	641
B.e.	Kirchhoff's Circuit Laws	643
Appendix C. Symmetries, Quarks, and Meson Masses		648
C.a.	Flavored Quarks	648
C.b.	Interactions of Quarks and Antiquarks	650
C.c.	The Lie Algebra of $SU(3)$	652
C.d.	Pions, Kaons, and Etas	653
C.e.	A Reduced Symmetry Group	656
C.f.	Meson Masses	658

Appendix D. Representations and Hyperelastic Bodies	660
D.a. Hyperelastic Bodies	660
D.b. Isotropic Bodies	661
D.c. Application of Schur's Lemma	662
D.d. Frobenius-Schur Relations	664
D.e. The Symmetric Traceless 3×3 Matrices Are Irreducible	666
Appendix E. Orbits and Morse-Bott Theory in Compact Lie Groups	670
E.a. The Topology of Conjugacy Orbits	670
E.b. Application of Bott's Extension of Morse Theory	673
<i>References</i>	679
<i>Index</i>	683