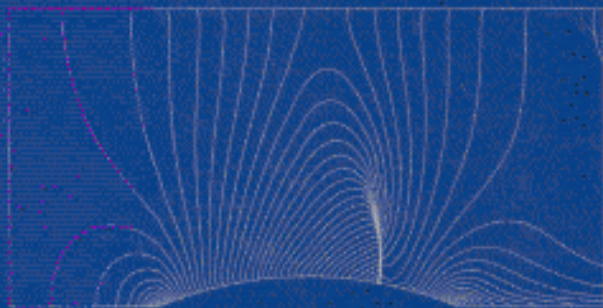


NUMERICAL MATHEMATICS
AND SCIENTIFIC COMPUTATION

Mathematical and Computational Methods for Compressible Flow

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