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# A HANDBOOK OF REAL VARIABLES

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WITH APPLICATIONS  
TO DIFFERENTIAL EQUATIONS  
AND FOURIER ANALYSIS

BIRKHAUSER

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