

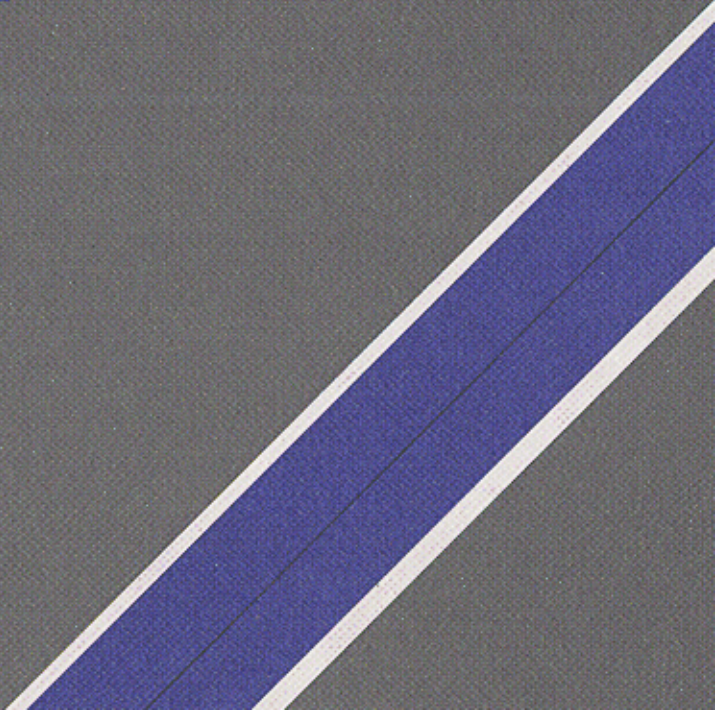
Cambridge studies in advanced mathematics

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# **Multidimensional Real Analysis I**

## **Differentiation**

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