

$$\int_0^{\infty} e^{-x} \ln x \, dx = -\gamma$$

IRRESISTIBLE INTEGRALS

Symbolics, Analysis and
Experiments in the Evaluation of Integrals

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$$\int_0^1 \ln \Gamma(x) \, dx = \ln \sqrt{2\pi}$$

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