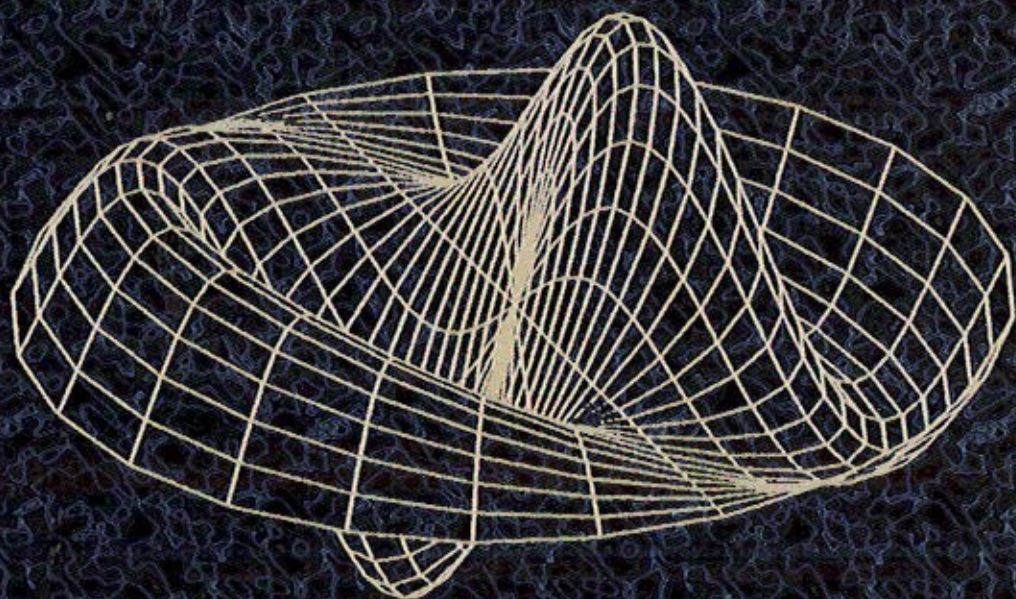


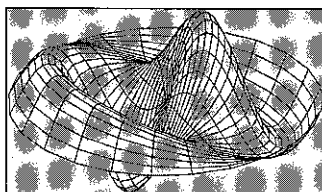
# Applied Partial Differential Equations

An Introduction



Alan Jeffrey

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