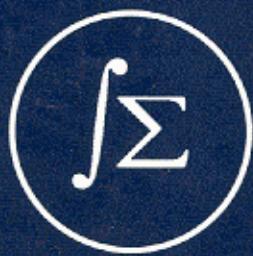


*Analytical Methods
and
Special Functions*

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**Methods of the
Theory of
Generalized
Functions**

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