

Marek Capiński and Ekkehard Kopp

Measure, Integral and Probability

$$\int_{\mathcal{R}^n} \nabla \cdot \vec{F} dV = \int_{\partial \mathcal{R}^n} \vec{F} \cdot \vec{n} d\sigma \iff \int_a^b dw = \int_a^b w$$

Second Edition $\neg(P \vee Q) = \neg P \wedge \neg Q$

$$|\langle \chi, \gamma \rangle| \leq \|\chi\| \|\gamma\|$$

$$\delta_g = \frac{1}{|G|} \sum_{x \in G} x_i(g) \overline{x_j(g)} = \frac{1}{|G|} \sum_{k \in G} k_i x_j(g) \overline{x_k(g)}$$

$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$



Springer

$$\int_a^b f(t) dt = F(b) - F(a)$$

S

SPRINGER

U

UNDERGRADUATE

M

MATHEMATICS

S

SERIES

Contents

1. Motivation and preliminaries	1
1.1 Notation and basic set theory	2
1.1.1 Sets and functions	2
1.1.2 Countable and uncountable sets in \mathbb{R}	4
1.1.3 Topological properties of sets in \mathbb{R}	5
1.2 The Riemann integral: scope and limitations	7
1.3 Choosing numbers at random	12
2. Measure	15
2.1 Null sets	15
2.2 Outer measure	20
2.3 Lebesgue-measurable sets and Lebesgue measure	26
2.4 Basic properties of Lebesgue measure	35
2.5 Borel sets	40
2.6 Probability	45
2.6.1 Probability space	46
2.6.2 Events: conditioning and independence	46
2.6.3 Applications to mathematical finance	49
2.7 Proofs of propositions	51
3. Measurable functions	55
3.1 The extended real line	55
3.2 Lebesgue-measurable functions	55
3.3 Examples	59
3.4 Properties	60
3.5 Probability	66

3.5.1	Random variables	66
3.5.2	σ -fields generated by random variables	67
3.5.3	Probability distributions	68
3.5.4	Independence of random variables	70
3.5.5	Applications to mathematical finance	70
3.6	Proofs of propositions	73
4.	Integral	75
4.1	Definition of the integral	75
4.2	Monotone convergence theorems	82
4.3	Integrable functions	86
4.4	The dominated convergence theorem	92
4.5	Relation to the Riemann integral	97
4.6	Approximation of measurable functions	102
4.7	Probability	105
4.7.1	Integration with respect to probability distributions	105
4.7.2	Absolutely continuous measures: examples of densities	107
4.7.3	Expectation of a random variable	114
4.7.4	Characteristic function	115
4.7.5	Applications to mathematical finance	117
4.8	Proofs of propositions	119
5.	Spaces of integrable functions	125
5.1	The space L^1	126
5.2	The Hilbert space L^2	131
5.2.1	Properties of the L^2 -norm	132
5.2.2	Inner product spaces	135
5.2.3	Orthogonality and projections	137
5.3	The L^p spaces: completeness	140
5.4	Probability	146
5.4.1	Moments	146
5.4.2	Independence	150
5.4.3	Conditional expectation (first construction)	153
5.5	Proofs of propositions	155
6.	Product measures	159
6.1	Multi-dimensional Lebesgue measure	159
6.2	Product σ -fields	160
6.3	Construction of the product measure	162
6.4	Fubini's theorem	169
6.5	Probability	173

6.5.1	Joint distributions	173
6.5.2	Independence again	175
6.5.3	Conditional probability	177
6.5.4	Characteristic functions determine distributions	180
6.5.5	Application to mathematical finance	182
6.6	Proofs of propositions	185
7.	The Radon–Nikodym theorem	187
7.1	Densities and conditioning	187
7.2	The Radon–Nikodym theorem	188
7.3	Lebesgue–Stieltjes measures	199
7.3.1	Construction of Lebesgue–Stieltjes measures	199
7.3.2	Absolute continuity of functions	204
7.3.3	Functions of bounded variation	206
7.3.4	Signed measures	210
7.3.5	Hahn–Jordan decomposition	216
7.4	Probability	218
7.4.1	Conditional expectation relative to a σ -field	218
7.4.2	Martingales	222
7.4.3	Doob decomposition	226
7.4.4	Applications to mathematical finance	232
7.5	Proofs of propositions	235
8.	Limit theorems	241
8.1	Modes of convergence	241
8.2	Probability	243
8.2.1	Convergence in probability	245
8.2.2	Weak law of large numbers	249
8.2.3	The Borel–Cantelli lemmas	255
8.2.4	Strong law of large numbers	260
8.2.5	Weak convergence	268
8.2.6	Central limit theorem	273
8.2.7	Applications to mathematical finance	280
8.3	Proofs of propositions	283
	Solutions	287
	Appendix	301
	References	305
	Index	307