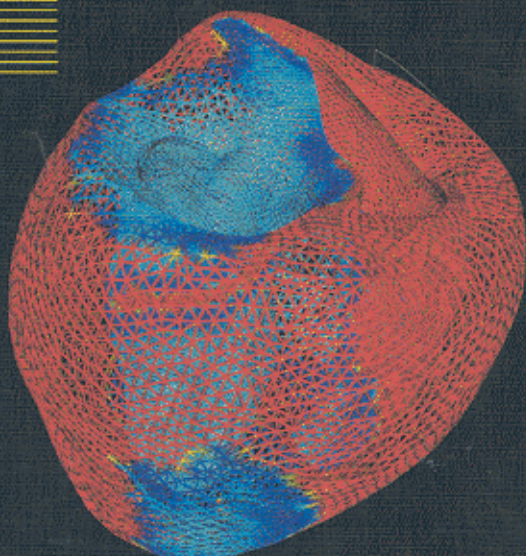


Texts in Computational Science
and Engineering

1



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Computational Partial Differential Equations

Numerical Methods
and Diffpack Programming

Second Edition



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