Super-resolution Image Restoration Algorithms Based on
Orthogonal Discrete Wavelet Transform

LIU Yang-yang,  JIN Wei-qi
(Department of Optical Engineering, School of Information Science Technology,
Beijing Institute of Technology, Beijing 100081, China)

1.Introduce

Recently, super-resolution image restoration theory develops rapidly. If the image degradation process is not
reversible, super-resolution image restoration algorithms might use a priori limited degradation parameters, under the
condition that the image low frequency information in frequency-pass bands can be restored, to restore blurred
images by restoring high frequencies beyond the cut-off frequency. Therefore image details can be retrieved so much
that the restored image is greatly close to the original object[1,2]. It is practical to image reconstruction.

Poisson-Luck-Richardson super-resolution image restoration algorithm based on Bayes statistical analysis theory
supposes that the image has Poisson distribution, and adopts Maximum-Likelihood theory to estimate blurred image,
as follows[3]:

\[
\hat{f}(x, y) = \frac{g(x, y)}{h(x, y) * \hat{f}(x, y)^n} \oplus h(x, y)
\]

(1)

where \( g \) is the degradation image, \( h \) is a point spread function of imaging system, \( * \) is a convolution parameter, \( n \) is
the order number of iterative, \( \oplus \) is a correlation operator. The algorithm has used a priori information about the object,
is applicable to linear and non-linear imaging models with a unique solution when noise is not severe. But the
restoration process may produce some ringing fringes, and the restored results are not ideal for high frequency loss
and distortion in relatively noisy images. Therefore images with ordinary signal to noise ratio (SNR) of 10–20
couldn’t be recovered well by the algorithm in practice.

In considering that the ODWT (for short ODWT) has the capability of indicating local features of a signal and
concentrating the signal power to a few coefficients in wavelet transform domain, it is introduced here and combined
with MPML to resolve the problem of low SNR image restoration.

2.Super-resolution Image Restoration Algorithms Based on ODWT

several new super-resolution image restoration algorithms based on ODWT are proposed, by using ODWT and
generalized cross validation, and combining with Luck-Richardson super-resolution image restoration algorithm (LR)
and Luck-Richardson algorithm based on Poisson-Markov model (MPML). Here is how:

2.1 LR algorithm Based on ODWT

The degradation process of discrete images is
\[ g(i, j) = h(i, j) * f(i, j) + \varepsilon(i, j) \quad (i, j = 1, \ldots, N) \quad (2) \]

where \( f(i, j) \) is the original image, \( \varepsilon \) is noise, \( (x, y) \) are 2D variables in space domain.

By formula (1), LR is updated as:

\[
f(i, j)^{n+1} = f(i, j)^n \left\{ \frac{h(i, j) * f(i, j)}{h(i, j) * f(i, j)^n} \right\} \oplus h(i, j) - \alpha \frac{\partial}{\partial f_{ij}} U(f^n) \quad (3)
\]

Assume that \( \varepsilon \) is a stationary stochastic uncorrelated noise with zero mean and variance \( \sigma^2 \). Formula (2) can be expressed with ODWT[^4] as

\[ Y = X + V \quad (4) \]

where \( X = W[h * f], V = W[\varepsilon], Y = W[g], \) and \( W \) is a 2D orthogonal wavelet transform matrix.

According to different characteristics of respective high frequency subbands, a suitable threshold \( \delta \) can be chosen to save the existed available signal remove or reduce noise influence on image, thus the noise effect in MPML may be reduced.

A thresholding operation \( T_\delta \) to wavelet transformed image \( Y \) yields:

\[ Y_\delta = T_\delta[Y] \quad (5) \]

All inverse wavelet transform will give the result:

\[ g_\delta = W^{-1}[Y_\delta] = W^{-1}\{T_\delta[W[g]]\} \quad (6) \]

The new algorithm from updated LR is:

\[
f(i, j)^{n+1} = f(i, j)^n \left\{ \frac{W^{-1}[Y_\delta(i, j)]}{h(i, j) * f(i, j)^n} \right\} \oplus h(i, j) \quad (7)
\]

### 2.2 MPML algorithms Based on ODWT

Poisson-Luck-Richardson super-resolution image restoration algorithm with Markov constraint (MPML) based on Bayes statistical analysis theory supposes that the image has Poisson and Markov distribution, and adopts Maximum-Likelihood theory to estimate blurred image, as follows[^3].

\[
f_\delta^{n+1} = f_\delta^n \left[ \left( \frac{g_{\delta}}{(h * f^n)_\delta} \right) \oplus h_{\delta} - \alpha \frac{\partial}{\partial f_{\delta}} U(f^n) \right]^p \quad (8)
\]

where \( U(f) \) is the energy function in Gibbs distribution, namely Markov constraint, the regularization self-adaptive parameter \( \alpha \) is usually adjusted synchronously with every iterative step, and two step-controlling coefficients \( K_1, K_2 \) which only could be confirmed by the respective input measured image are needed[^3], to implement its self-adaptive process for constraining weight unbalance part in iterative process, and \( p \) is a controlling coefficient for controlling the algorithm’s convergence and iterative speed.
The new algorithm from updated MPML is: the algorithm of MPML operation after wavelet transform (W-MPML) and the algorithm of MPML operation before wavelet transform (MPML-W), where W-MPML is:

\[ f_{ij}^{n+1} = f_{ij}^{n} \left[ \left( \frac{g_{\delta}}{h * f^{n}_{ij}} \right) \gamma + h_{ij} - \alpha \frac{\partial}{\partial f_{ij}^{n}} U(f^{n}) \right] \]  
(9)

and MPML-W is:

\[ f_{ij}^{n+1} = \{ f_{ij}^{n} \left[ \left( \frac{g_{\delta}}{h * f^{n}_{ij}} \right) \gamma + h_{ij} - \alpha \frac{\partial}{\partial f_{ij}^{n}} U(f^{n}) \right] \}^{\delta} \]  
(10)

According to simulating experiments for practical images, restoration results and evaluating parameters of MPML-W and W-MPML are obviously better than MPML itself, but the degraded images are still recovered poorly.

3. Single Operation Algorithms Based on ODWT

3.1 Single Operation WLR Algorithm

According to reference [5] proved, under the condition of no noise or noise could be omitted, there is one region of \( S = \{ f : \| f - \hat{f}^{1} \| < \delta \} \), only when LR the first iterative vector \( \hat{f}^{1} \) included in \( S \), LR have the ability of converge at immovable point that is LR is converged.

From the physics meaning, referring to lower SNR (larger noise) problem, LR and WLR both satisfied the condition of convergence immovably point, namely iterative of two is ineffective, which can also be seen in practical experiments. Whereas WLR didn’t begin iterative operation, namely when \( n \) in formula.7 equals 1, restoration results is much better than degradation and restored images by LR and WLR algorithms, and even operation time is fewer than WLR. And the MSE value of reconstructed image by W-MPML and MPML-W along with iterative times is presented as Fig.1.

Therefore the single operation WLR algorithm is further proposed for improving low SNR image resolution better and reducing process operation work, even operation time, as follows (Single Wavelet LR, SWLR):

\[ \hat{f}(i, j) = g_{\delta}(i, j) \left[ \left( \frac{g_{\delta}(i, j)}{h(i, j) * g_{\delta}(i, j)} \right) \gamma \right. \]  
(11)

where \( \hat{f}(i, j) \) is restored image. SWLR algorithm give attention to the de-blurring advantages in LR, at the same time, they utilize more the superior de-noising capability of ODWT instead of iterative process and even time in LR.
SWLR algorithm is instantaneous under 10～20db of low SNR image restoration project for application.

3.2 Single Operation W-MPML and MPML-W Algorithm

From the physics meaning, referring to lower SNR (larger noise) problem, the restriction from the regulation parameter $\alpha$ and wavelet transform is destroyed by too much iterative in W-MPML and MPML-W, which can also be seen in practical experiments. Whereas W-MPML and MPML-W didn’t begin iterative operation, namely when $n$ in formula.9 and 10 equals 1, the constraint factors reach the optimizing match with each other, so restoration results is both better than degradation and restored images by W-MPML and MPML-W algorithm, and even operation time is much fewer than W-MPML and MPML-W. And the MSE value of reconstructed image by W-MPML and MPML-W along with iterative times is presented as Fig.2 and Fig.3.

Therefore the single operation W-MPML and MPML-W algorithms is further proposed, the single operation W-MPML algorithm can be expressed as:

$$
\hat{f}_{ij} = g_{ij} \left[ \frac{g_{ij}}{(h * f)_{ij}} \right] \hat{h}_{ij} - \alpha \frac{\partial}{\partial f_{ij}} U(g_{ij})^\delta
$$

The single operation MPML-W algorithm is how:

$$
\hat{f}_{ij} = {f_{ij}} \left[ \frac{g_{ij}}{(h * g)_{ij}} \right] \hat{h}_{ij} - \alpha \frac{\partial}{\partial f_{ij}} U(g)^\delta
$$

SW-MPML and MPML-SW algorithms give attention to the de-blurring advantages, and extract the energy function under Markov distribution in MPML , at the same time, they utilize more the superior de-noising capability of ODWT instead of $\alpha$, with timesaving $\alpha$ as one constant fit of common to omit iterative process and even time. The SW-MPML and MPML-SW algorithms are instantaneous under 10～20db of low SNR image restoration project.

Fig.2  The MSE value of reconstructed image by W-MPML along with iterative times

Fig.3  The MSE value of reconstructed image by MPML-W along with iterative times

However the validity of SWLR, even SW-MPML and MPML-SW is decided by the optimal threshold $\delta$ in the respective high frequency subbands, so how to choose the optimal threshold $\delta$ is the key for new proposed algorithms.
4. Threshold Choosing

It is difficult for practical application to use the Mean Square Error function and other more complex parameters in evaluating the optimal threshold \( \delta \), so the Generalized Cross Validation theory (GCV) is introduced to resolve this problem here \([6,7]\).

4.1 Threshold Derivative Matrix

According to the proposed soft-threshold by reference \([7]\), the updated new matrix \( T_\delta \) is:

\[
Y_\delta(i, j) = T_\delta[Y] = \begin{cases} 
0 & |Y(i, j)| < \delta \\
Y(i, j) - \delta & |Y(i, j)| \geq \delta 
\end{cases} \quad (i, j = 1, \ldots, N) \tag{14}
\]

The cells \( t'(m, n) \) of the threshold derivative matrix \( T' \) could be represented as:

\[
t'(m, n) = \frac{\partial Y_\delta(i, j)}{\partial Y(k, l)} = \begin{cases} 
0 & m \neq n \\
0 & |Y(i, j)| < \delta \\
1 & |Y(i, j)| \geq \delta & m = n
\end{cases} \tag{15}
\]

where \( k, l = 1, \ldots, N; \quad m = (i-1)N + j, \quad n = (k-1)N + l \).

4.2 Optimal Threshold Determined by Minimizing Generalized Cross Validation

An asymptotically optimal threshold is determined by minimizing Generalized Cross Validation, which is based on Ordinary Cross Validation and aimed at an unknown exact signal \([6]\). Assume that \( \tilde{Y}_i \) is a neighbours combination of \( Y_i \) \((i = 1, \ldots, N^2)\), not including \( Y_i \) itself, such as \( \tilde{Y}_i = \frac{1}{2}(Y_{i-1} + Y_{i+1}) \), so an optimal combination \( \tilde{Y}_i \) to minimize noise is possible. Similarly, assume that \( \tilde{Y}_{\delta i} \) is a neighbours combination of \( Y_{\delta i} \) \((i = 1, \ldots, N^2)\), then \( \tilde{Y}_{\delta i} = \tilde{Y}_i \) will turn out to be an interesting choice. “Ordinary Cross Validation” (OCV) is:

\[
OCV(\delta) = \frac{1}{N^2} \sum_{i,j=1}^{N} [Y(i, j) - \tilde{Y}_{\delta i}(i, j)]^2 \tag{16}
\]

For too small values of \( \delta \), the difference \( Y(i, j) - \tilde{Y}_{\delta i}(i, j) \) is dominated by noise, while for large values of \( \delta \), the signal itself is too much distorted. So the \( \delta \) of minimized OCV corresponds to the best compromise between image fitting and smoothness.
Formula (16) is transformed because of unidentified $\tilde{Y}_\delta$ under unknown $\delta^6$, So OCV is represented as:

$$OCV(\delta) \approx \frac{1}{N^2} \sum_{i,j=1}^{N^2} \frac{[Y(i, j) - \tilde{Y}_\delta(i, j)]^2}{[1 - \ell'(m, n)]^2}$$

(17)

However this cannot be used in practical computations, since $\ell'(m, n)$ is 0 or 1. Therefore some kind of mean value for $[1 - \ell'(m, n)]$ is taken, this gives the formula of the so called “Generalized Cross Validation” [6]:

$$WGCV(\delta) = \frac{1}{N^2} \frac{\|Y - \tilde{Y}_\delta\|^2}{\text{trace}(I - T')}$$

(18)

If $\delta^* = \arg\min MS E(\delta)$ and $\tilde{\delta} = \arg\min GCV(\delta)$, reference [6] proves that for $N \rightarrow \infty$, both minimizers yield a result of the same quality:

$$\frac{EMSE(\tilde{\delta})}{EMSE(\delta^*)} \rightarrow 1$$

(19)

That is why the $\delta$ obtained from GCV is asymptotically optimal. As for the estimating solution process of $\delta$, the value range of $\delta$ is positive value in wavelet decomposition coefficients, namely the value range of $Y$, with the WGCV computation results as Fig.3, and $\delta$ don’t need be computed exactly. So the stationary point $\delta$ corresponding to minimized WGCV ($\delta$) is chosen as optimal $\delta$.

5 Experimental Results

5.1 Super-resolution Image Restoration Algorithms Based on ODWT

The closeness of restoration to original image is evaluated by Mean Square Error (MSE), Peak SNR, and
improved SNR. A 256×256 “Lena” image (SNR=15) is restored as Fig.5, where MPML reconstructing advantage when a image of SNR=30 is presented as (c).

5.2 Single Operation Algorithms Based on ODWT

“Lena” restoration effects by every Single Operation Algorithms is presented as Fig.6.
A comparison among MSE, PSNR, ISNR is shown in Table 1:

<table>
<thead>
<tr>
<th>image</th>
<th>SNR(db)</th>
<th>algorithms</th>
<th>MSE</th>
<th>PSNR</th>
<th>ISNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lena</td>
<td>15</td>
<td>Degradation image</td>
<td>0.0057</td>
<td>22.4413</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>L-R</td>
<td>0.0459</td>
<td>13.3819</td>
<td>-9.0594</td>
</tr>
<tr>
<td></td>
<td></td>
<td>WL-R</td>
<td>0.0387</td>
<td>14.1229</td>
<td>-8.3184</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SWL-R</td>
<td>0.0031</td>
<td>25.0864</td>
<td>2.6451</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Degradation image</td>
<td>0.0057</td>
<td>22.4413</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>MPML</td>
<td>0.2353</td>
<td>6.2838</td>
<td>-16.1575</td>
</tr>
<tr>
<td></td>
<td></td>
<td>W-MPML</td>
<td>0.0922</td>
<td>10.3527</td>
<td>-12.0886</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MPML-W</td>
<td>0.1474</td>
<td>8.3150</td>
<td>-14.1263</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SW-MPML</td>
<td>0.0053</td>
<td>22.7572</td>
<td>0.3159</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MPML-SW</td>
<td>0.0055</td>
<td>22.5964</td>
<td>0.1551</td>
</tr>
</tbody>
</table>

6. Conclusion

To seek a way for resolving image restoration problem under lower SNR, the algorithm of LR after wavelet transform (WLR), MPML operation after wavelet transform (W-MPML), the algorithm of MPML operation before wavelet transform (MPML-W), and the single operation of WLR, W-MPML and MPML-W algorithms are proposed. After degradation image is “Symlets” wavelet transformed, an asymptotically optimal threshold is determined by minimizing Generalized Cross Validation theory, and high frequency subbands in each decomposition level are denoised with soft threshold processes to converge respectively to those with maximum SNR, when the method is incorporated with existed LR and MPML, details of original image, especially those with low SNR could be well recovered. SWLR, SW-MPML and MPML-SW are some operative algorithms proposed based on the method. According to the processing results of simulating and practical images, in comparison with LR and MPML, because of only one operation, under the guarantee of rapid and effective processing. They could also remove iterative operation time for up to hundreds times, as well as avoid the iterative operation of self-adaptive parameters in MPML, improve operating speed and precision. They are practical and instantaneous to low signal-noise-ratio image restoration.
References


