Representation of the Wigner distribution function for light beams passing through apertured optical systems

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ABSTRACT

By introducing a method that a hard-edged aperture function can be expanded into an approximate sum of complex Gaussian functions with finite numbers, the analytical expression of Wigner distribution function for a Gaussian beam passing through a cylindrical symmetric and paraxial \textit{ABCD} optical system with a hard-edged aperture is obtained. Numerical calculations show that the effect of an aperture on the Wigner distribution function is prominent. The analytical results are also compared with the integral calculation results and they show that this method of expanding a hard aperture into Gaussian functions with finite numbers is proper and ascendant. This method could also be extended to studying the Wigner distribution functions of other light beams passing through a paraxial \textit{ABCD} optical system with a hard-edged aperture.

Keywords: Wigner distribution function, Gaussian beam, aperture, \textit{ABCD} optical systems

1. INTRODUCTION

The Wigner distribution function, originally discovered and used in the context of quantum mechanics\textsuperscript{1}, has become a powerful tool in the description of both coherent and partially coherent beams and their passage through first-order systems\textsuperscript{2-8}. The Wigner function that corresponds to a Hermite-Gaussian mode has been known since the early days of quantum mechanics\textsuperscript{9} and it takes a simple closed-form in terms of Laguerre polynomials. Furthermore, the Wigner function that corresponds to a Laguerre-Gaussian mode is also given and takes a simple closed-form in terms of Laguerre polynomials\textsuperscript{10-11}. Recently, the Wigner distribution function of a circular or rectangular aperture was determined\textsuperscript{12}.

It is known that study on the propagation of the Wigner distribution function for a light beam passing through an optical system with a hard-edge aperture is a difficult and interesting problem, to the best of our knowledge, which has been paid little attention. In fact, a hard-edge aperture always exists in practically optical systems and can not be avoided, for instance, the finite size of a lens or an optical element. Therefore study on the behavior of light beams through the apertured optical systems would be of practical interest. In this paper, taking the incidence of a Gaussian beam as an example, we will study the propagation of its Wigner distribution function through a cylindrical symmetric and paraxial \textit{ABCD} optical system with a hard-edged aperture.

The approximate analytical representation of the Wigner distribution function for a Gaussian beam passing through an apertured optical system is derived. Taking its propagation through an apertured free space as an example, some numerical simulations are done. Finally, a simple conclusion is outlined.

2. APPROXIMATE ANALYTICAL REPRESENTATION OF THE WIGNER DISTRIBUTION FUNCTION FOR A GAUSSIAN BEAM PASSING THROUGH AN APERTURED \textit{ABCD} OPTICAL SYSTEM

In one-dimensional case, the Wigner distribution function that corresponds to an optical field amplitude \(E(x)\) at a chosen transverse plane is defined by\textsuperscript{1-2}

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\[ W(x, u) = \int_{-\infty}^{+\infty} E(x + \frac{x^2}{2}) E^*(x - \frac{x^2}{2}) \exp(-2\pi ix^2u) dx \]  

(1)

where \( x \) denotes the position variable, \( u \) means the spatial-frequency variable in phase space.

It is well known that the fundamental mode Gaussian beam, which is a special solution of wave equation under slowly varying amplitude approximation, is a characteristic and interesting one in laser optics. Let us consider a one-dimensional Gaussian beam in Cartesian coordinate domains given by

\[ E_i(x_i) = \exp(-\frac{x_i^2}{w^2}) \]  

(2)

is incident upon a strip hard-edged aperture followed by an \( ABCD \) optical system, where \( w \) is the beam waist size.

The corresponding Wigner distribution function of the input Gaussian beam is obtained as

\[ W_i(x_i, u_i) = (\frac{\pi}{\alpha})^{1/2} \exp(\frac{\beta^2}{\alpha}) \exp(-\frac{2\pi w^2}{\alpha^2} x_i^2) \exp(-2\pi i u_i^2 w^2) \]  

(3)

where \( \alpha = 1/(2\omega^2) \) and \( \beta = i\omega_i \).

Introducing the strip hard-edged aperture function

\[ A_p(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases} \]  

(4)

where \( a \) denotes the half width of the hard-edged aperture. Generally, the aperture function can be expanded as the sum of complex Gaussian functions with finite numbers and is given by

\[ A_p(x) = \sum_{n=1}^{N} A_n \exp(-\frac{B_n}{a^2} x^2) \]  

(5)

where \( A_n \) and \( B_n \) are the expansion and Gaussian coefficients, respectively, which could be obtained by optimization-computation directly\(^{13} \). However, it should be noted that Eq. (5) is only an approximate expression for the hard-edged aperture function. It has been shown that the larger the \( N \) is and the higher the simulation efficiency is\(^ {14 \text{--} 15} \).

In the paraxial approximation, most of optical elements and optical systems can be characterized by \( ABCD \) transfer matrix and its determinant of the matrix is a constant for a linear optical system\(^ {16} \). According to Collins’ diffraction integral formula in Fresnel approximation\(^ {17} \), the output field distribution for a Gaussian beam passing through an apertured \( ABCD \) optical system is obtained as follows

\[ E(x_2) = \frac{-ik}{2dB} \int_{-\infty}^{+\infty} E_i(x_i) A_p(x_i) \exp\left(\frac{ik}{2B} (Ax_i^2 - 2x_i x_2 + Dx_i^2)\right) dx_i \]  

(6)

where \( L_0 \) is a constant eikonal function, \( k \) is the wavenumber. Substituting Eqs. (5) and (2) into (6), we obtain the output field distribution as

\[ E(x_2) = \frac{-ik}{2B} \exp(ikL_0) \sum_{n=1}^{N} A_n \exp\left(\frac{ikD}{2B} (x_2^2)(a')^{-1/2} \exp\left(\frac{\beta^2}{\alpha'}\right)\right) \]  

(7)

where

\[ \alpha' = \frac{1}{w^2} + \frac{B_n}{a^2}, \quad \beta = \frac{ikx_2}{2B} \]  

(8)

Substituting Eq. (7) into (1), after performing the tedious integration, we obtain the Wigner distribution function at the output plane as

\[ W_2(x_2, u_2) = \frac{-ik\pi^{1/2}}{2B} \sum_{n=1}^{N} \sum_{p=1}^{N} A_n A_p^* (\alpha \alpha')^{-1/2} \exp\left(-4\alpha(\alpha') x^2\right) \exp\left(\frac{\beta^2}{\alpha'}\right) \]  

(9)

where
\[ \alpha^* = \frac{k^2}{16B^2} \left( \frac{1}{\alpha} + \frac{1}{\alpha'} \right) \]  

(10)

\[ \beta^* = \frac{k^2x_2^*}{8B^2} \left( \frac{1}{\alpha} - \frac{1}{\alpha'} \right) - \frac{ikDx_2^*}{2B} + \pi u_z \]  

(11)

For \( B = 0 \) (corresponding to image forming systems), Eq. (6) seems divergent. Actually, this is not the case and the output field distribution becomes

\[ \sum_{n=1}^{N} \alpha_n A_n^* \left( \frac{4\alpha^2}{\alpha} \right)^{1/2} \exp \left( \frac{\beta^2}{\alpha} \right) \]  

(12)

where \( A \) means the transverse magnification factor, then the corresponding Wigner distribution function at the output plane becomes

\[ W_2(x_2,u_2) = \sum_{n=1}^{N} A_n A_n^* \left( \frac{4\alpha^2}{\alpha} \right)^{1/2} \exp \left( \frac{\beta^2}{\alpha} \right) \]  

(13)

where

\[ \alpha = \frac{1}{2w^2A^2} + \frac{B_n + B_n'}{4a^2A^2} \]  

(14)

\[ \beta = \frac{(B_n - B_n')x_2^*}{2a^2A^2} - \frac{ikCn^2}{2A} + \pi u_z \]  

(15)

### 3. NUMERICAL SIMULATIONS

Eqs. (9) and (13) are the main results of this paper, which are the approximate analytical propagation expressions of a Gaussian beam through an apertured cylindrical symmetric and paraxial AB\(CD\) optical system. For the simplicity, let us consider the case of a hard-edged aperture followed by a free space, which is written in terms of the ray transfer matrix elements \( A=D=1, B=z, C=0 \). where \( z \) denotes the distance propagating in free space. From Eq. (9) its Wigner distribution function at the output plane becomes

\[ W_2(x_2,u_2) = -\frac{ik}{2z} \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha A_n A_n^* \left( \frac{4\alpha}{\alpha} \right)^{1/2} \exp \left( -\frac{4\alpha x_2^*}{\alpha} \right) \left( \frac{\pi}{\alpha'} \right)^{1/2} \exp \left( \frac{\beta^2}{\alpha} \right) \]  

(16)

where

\[ \alpha' = \frac{1}{w^2} + \frac{B_n}{2z}, \quad \alpha^* = \frac{k^2}{16z^2} \left( \frac{1}{\alpha} + \frac{1}{\alpha'} \right), \quad \beta' = \frac{k^2x_2}{8z^2} \left( \frac{1}{\alpha} - \frac{1}{\alpha'} \right) - \frac{ikx_2}{2z} + \pi u_z \]  

(17)

In the numerical calculation, \( f = 200 \text{mm} \), the wavelength \( \lambda = 1.06 \mu \text{m} \), the waist size of Gaussian beam \( w_0 = 0.5 \text{mm} \), the equivalent Fresnel number \( N_f = w_0^2/(\lambda f) \).

First, let us examine what extent the expansion of the Gaussian functions represented by Eq. (5) matches the hard aperture function. Fig. 1(a) shows the real and imaginary parts of Eq. (5), evaluated by the coefficients \( A_n \) and \( B_n \) with \( N=10 \) listed in Table 1 which are given by Wen and Breazeale. Fig. 1(b) shows the magnitude and phase of the Gaussian expansion. One can see at the figure that the errors reach \( \sim 5-6\% \) and even \( \sim 12\% \) near the aperture edge. A hard-edged aperture can well be expanded into the sum of 10 or more complex Gaussian functions.

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Table 1. The coefficients $A_n$ and $B_n$ with $N=10$.

<table>
<thead>
<tr>
<th>N</th>
<th>$A_n$</th>
<th>$B_n$</th>
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<tbody>
<tr>
<td>1</td>
<td>11.428+0.95175i</td>
<td>4.0697+0.22726i</td>
</tr>
<tr>
<td>2</td>
<td>0.06002-0.08013i</td>
<td>1.1531-20.933i</td>
</tr>
<tr>
<td>3</td>
<td>-4.2743-8.5562i</td>
<td>4.4608+5.1268i</td>
</tr>
<tr>
<td>4</td>
<td>1.6576+2.7015i</td>
<td>4.3521+14.997i</td>
</tr>
<tr>
<td>5</td>
<td>-5.0418+3.2488i</td>
<td>4.5443+10.003i</td>
</tr>
<tr>
<td>6</td>
<td>1.1227-0.68854i</td>
<td>3.8478+20.087i</td>
</tr>
<tr>
<td>7</td>
<td>-1.0106-0.26955i</td>
<td>2.528-10.31i</td>
</tr>
<tr>
<td>8</td>
<td>-2.5974+3.2202i</td>
<td>3.3197-4.8008i</td>
</tr>
<tr>
<td>9</td>
<td>-0.1484-0.31193i</td>
<td>1.9002-15.82i</td>
</tr>
<tr>
<td>10</td>
<td>-0.2085-0.23851i</td>
<td>2.634+25.009i</td>
</tr>
</tbody>
</table>

Fig. 1. (a) Real and imaginary parts of the Gaussian expansion for the aperture function, (b) magnitude and phase of the Gaussian expansion.

Figs. 2 and 3 show the normalized Wigner distribution functions at the output plane with $N_f=1.0$ of a Gaussian beam passing through an apertured free space, calculated by the analytical expression of Eq. (16) and the direct integral formula, respectively. Figs. 4 and 5 show the normalized Wigner distribution functions at the output plane with $N_f=0.1$. By comparing these two groups of figures, we find that the results obtained by the approximate analytical expressions and by the direct integral formula are in accordance with each other very well in the centre of the figures but some small deviations away from the centre. Figures show that the effect of a hard-edged aperture on the Wigner distribution function is prominent in the case when $a/w_0=1$ and the diffraction effect of the hard-edged aperture can be neglected in the case when $a/w_0=3$. By numerical calculations, we find that there exists great difference in the calculation efficiency of the approximate analytical expressions and of the direct integral formula. The former is far higher than that of latter, especially in the case when $a/w_0=3$ and $N_f=0.1$. So this method of expanding a hard-edged aperture into Gaussian functions with finite numbers is proper and ascendant.
Fig. 2. The normalized Wigner distribution functions at the output plane of a Gaussian beam passing through an apertured free space calculated by approximate analytical expression of Eq. (16). Where, the equivalent Fresnel number $N_f = \frac{1}{fN}$. (a) $a / \omega_0 = 1$; (c) $a / \omega_0 = 3$; (b) and (d) are the projections of (a) and (c), respectively.
Fig. 3. The same as Fig. 2 but calculated by the direct integral formula.

Fig. 4. The normalized Wigner distribution functions at the output plane of a Gaussian beam passing through an apertured free space calculated by approximate analytical expression of Eq. (16). Where, the equivalent Fresnel number $N_f = 0.1$. (a) $a/w_0 = 1$; (c) $a/w_0 = 3$; (b) and (d) are the projections of (a) and (c), respectively.
4. CONCLUSIONS

In conclusion, the approximate analytical representation of Wigner distribution function for a Gaussian beam passing through an apertured $ABCD$ optical system are obtained and some numerical calculations are also illustrated for its application. It provides a simple and powerful way to treat the propagation of Wigner distribution function in apertured $ABCD$ optical systems. It should be pointed out that only real-valued $ABCD$ optical systems are treated in this paper, in fact, the results could be straightforwardly extended to complex $ABCD$ optical systems. For example, Gaussian aperture could be regarded as a complex optical element. This method of expanding hard-edged aperture into Gaussian beams with finite numbers used in this paper could also be applied to treating Wigner distribution functions for other light beams, such as Hermite-Gaussian beams, Laguerre-Gaussian beams, Hermite-sinusoidal-Gaussian beams, etc., passing through apertured $ABCD$ optical systems.

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