Experimental Determination of Fractal Dimension of the Ordinary Fractal Patterns with Optical Fractional Fourier Transform*  
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ABSTRACT  
By means of experimental technique of optical fractional Fourier transform, we have determined the fractal dimension of a regular ordinary fractal pattern to demonstrate the feasibility of this approach. Experimental results show that optical fractional Fourier transform is a practical method to analyze the ordinary fractal patterns.  

Keywords: optical fractional Fourier transform, regular fractal, Dimension  

1. INTRODUCTION  
The extension of ordinary Fourier transform to the fractional Fourier transform (FRFT) was first accomplished by Namias in 1980, he gave a complete mathematical definition and discussed the Eigenfunction of the transform. Using a system of lenses Lohmann first successfully realized the FRFT optically and designed the single lens mode and double lens mode for the realization of the FRFT of continuously variable fractional order 1.  

The concept of fractals was first proposed by Mandelbrot 2. Theory of fractals was used to study the geometrical aspect of the self-similarity of certain inhomogeneous patterns, in conjunction with theories of dissipative structures and chaos, they became the powerful means for the study of complex phenomena of nature. The study of fractals is very extensive and active now, becoming a frontier topic of contemporary nonlinear science.  

Alieva gave a theoretical analysis of optical FRFT of the regular fractals 3. Based on this theory we determined the fractal dimensions of some regular fractal patterns with optical FRFR for the first time. It demonstrates the feasibility of a new method for the determination of the fractal dimension of the regular fractals.  

2. THEORETICAL METHOD  
Fractal originally means fragmentary disordered structure with scaling symmetry or self-similarity, i.e., invariance with respect to dilation or contraction.  

Fractals may be divided into several kinds: the ordinary fractals, the self-affine fractals and multifractals 4. The scaling properties are isotropic for ordinary fractals. There also are a lot of patterns which have anisotropic scaling properties, i.e. different scaling properties along different directions, these are called the self-affine fractals. The scaling properties may be characterized by the function \( f(x) \) (where \( x \) is a \( n \)-dimensional arbitrary vector) which satisfy the following relations:  

\[
f(\lambda x) = \lambda^H f(x)
\]  

where \( \lambda \) is the scaling factor, \( H \) is the Hurst exponent. The local box dimension \( D \) of the ordinary fractal embedded in a \( d \)-dimensional Euclidean space is written as:  

\[
D = d - H
\]  

Ordinary fractals as well as the self-affine fractals may be further divided into two kinds: one is regular fractal with strict self-similarity constructed according to certain mathematical rules; while the other is irregular one with
self-similarity only in the statistical sense, these are mostly found in nature.
The FRFT may be regarded as an extended version of the ordinary FT and depends on a parameter $\alpha$ that can be interpreted as a rotation by the angle $\alpha$ in temporal frequency or spatial frequency planes. The order of FRFT may be varied continuously by gradually changing $\alpha$ from $\alpha = 0$ to $\alpha = \pi/2$. At $\alpha = \pi/2$, the FRFT returns to ordinary FT. The $n$-dimensional FRFT may be written as:

$$R^\alpha f(x) = F^\alpha_x(u) = \int_{-\infty}^{\infty} f(x)K^\alpha(x,u)dx$$

where $x$, $u$ are $n$-dimensional vectors

$$K^\alpha(x,u) = (2\pi i \sin \alpha)^{-n/2} \exp(i \frac{\alpha n}{2}) \exp \left\{ \frac{i}{2} \sin \alpha \left[ \cos \alpha (x^2 + u^2) - 2xu \right] \right\}.$$ (4)

According to the generalized similarity theorem for the FRFT, Eq.3 can be written as:

$$R^\alpha f(\lambda x) = (\gamma \lambda^{-1})^{n/2} \exp \left\{ \frac{i\pi}{2} (\alpha - \beta) \right\} \exp \left\{ \frac{i}{2} u^2 \cot \alpha (1 - \gamma^2 \lambda^{-2}) \right\} F_\beta(\gamma, u),$$ (5)

where $\gamma = \sin \beta (\lambda \sin \alpha)^{-1}$, $F_\beta(u)$ is the FRFT of $f(x)$ at angle $\beta$ such that

$$\cot \alpha = \lambda^2 \cot \beta.$$ (6)

The linearity of the transform and its scaling behavior Eq.3 lead to a relation for the squared modulus of the FRFT of $f(x)$ at angles $\alpha$ and $\beta$:

$$\left| F^\alpha_x(u) \right|^2 = \lambda^{-2H-n} \gamma^n \left| F_\beta(\gamma u) \right|^2,$$ (7)

where $\alpha, \beta \neq \pi m/2$, and $\beta = \frac{\pi}{2} - \alpha$, $\gamma = 1$.

From Eq. 5, it is known that $\alpha = \arctan \lambda^{-1}$, thus Eq.6 may be changed into:

$$\left| F^{\pi/2-\alpha}(u) \right|^2 = \left| \cot \alpha \right|^{2H+n} \left| F^\alpha_x(u) \right|^2.$$ (8)

To study optical FRFT of the ordinary fractals, at input plane $z = 0$, the light beam propagates along z-axis, the object is a regular ordinary fractal (Eq.1). Let input function is $\Psi(r_0, 0)$, thus 2-dimensional FRFT can be expressed as:

$$\Psi(r, z) = \int_{-\infty}^{\infty} \Psi(r_0, 0)K^\alpha(r_0, r)dr = F^\alpha_x(r).$$ (9)

Since angle $\alpha$ is proportional to $z$, i.e., $\alpha = gz$, the light intensity at $z$ plane $I(r, z)$ is equal to the squared modulus of 2-dimensional FRFT at that place

$$I(r, z) = I(r, \alpha) = \left| F^\alpha_x(r) \right|^2.$$ (10)

According to Eq.7 and Eq.9, finally we get:

$$I(r, \alpha) = \left| \cot \alpha \right|^{2H+2} I(r, \alpha).$$ (11)

We may utilize Eq.11 to design experimental program to determine the Hurst exponent. Then use Eq.2 to obtain the dimension of ordinary fractals. The practical method is to measure the light intensity of the FRFT at complementary positions $\alpha$ and $\beta(\beta = \frac{\pi}{2} - \alpha)$. 

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3. EXPERIMENTAL RESULTS AND DISCUSSIONS

In our experiment, single lens mode for optical FRFT is adopted as shown in Fig. 1. Light source is a He-Ne laser ($\lambda = 632.8\text{nm}, 15\text{mw}$) a pair of polaroids are used as the polarizer and the analyzer to control output light intensity. After a beam expander with pin-hole filter $G$ and a screen for blocking stray light $K$, a lens $L_1 (f=29.3\text{cm}, D=7.5\text{cm})$ is used to obtain parallel light beam. $S_1$ is the stage for the regular fractal specimen, $L_2 (f=58.9\text{cm}, D=13.5\text{cm})$ is a lens for getting FRFT spectra, $S_2$ is the stage for reception of the image of FRFT, through a CCD array the image is transported into a computer for processing.

We use a regular ordinary fractal pattern with known fractal dimension $D = 1.8928, H = 0.1072, d = 2$, shown in Fig. 2 as specimen to verify feasibility of theoretical formula Eq. 11. Changed $\alpha$ and utilized CCD array, we obtain a series images of specimen, shown in Fig. 3. Then we used the software named Photoshop to analyse the images. First, we managed to determine proportion between the complementary images. We find two obvious correspond dots in each complementary images, then measure the distance between the two dots, The proportion of the distances is what we need. Second, using the proportion, we take corresponding little plot of image in complementary images into studying, shown in Fig. 4. We utilized the center of complementary images to find corresponding parts of the images, because we put the specimen in the optical axis, the place of center of images we get in the experiment dose not change with $\alpha$, and it can be used as the reference dot. Using the tool called histogram of Photoshop, we obtain the intensity distribution at two complementary positions $I(r, \alpha)$ and $I(r, \frac{\pi}{2} - \alpha)$. Finally, the intensity ratio for complementary positions were measured.

![Fig.1 Schematic diagram for the experimental set-up](image1)

![Fig.2 The specimen used in the experiment](image2)
Fig. 3 The patterns of optical fractional Fourier transform of some regular fractal patterns
Fig. 5 give four straight lines fitted for experimental data by the method of least squares, the abscissa represents \( \ln \cos \alpha \), the ordinates represents \( \ln \left( \frac{I(r, \frac{\pi}{2} - \alpha)}{I(r, \alpha)} \right) \). The slope equals to \( H' = 2H + 2 \), then the Hurst exponent \( H \) is determined. Using Eq. 2, we can get the dimension \( D \) for the specimen. The experimented errors of the values of fractal dimension thus determined are within 6% of its true value. It shows the feasibility of this method for the experimental determination of fractal dimension of the regular fractal patterns. The principal sources of experimental error are due to following points: the error in the determination of complementary pairs of positions for the measurement of intensity corresponding to \( \alpha \) and \( \frac{\pi}{2} - \alpha \); since \( \ln \cot \alpha > 0 \), so \( \alpha < 45^\circ \), if we choose the value of \( \alpha \) to be too small, then the distance between lens and the object is too small to be accurately measured and at the same time the distance between the complementary pairs of positions is too large for the FRFT spectra to be matched. In our experiment the values of \( \alpha \) are chosen within the range \( 35^\circ < \alpha < 45^\circ \), in order to minimize the error. The data for several pairs of complementary positions are plotted, and a series of straight lines were obtained from least-square fitting, the fractal dimensions are determined from the slopes of these straight lines.
4. CONCLUSION

This paper gives the result of an experimental research on optical FRFT of some regular ordinary fractal patterns, based on the theory of optical FRFT of regular fractal patterns. The dimension of some regular ordinary fractal patterns are determined experimentally. Experimental results show that this method can determine the fractal dimension of some regular and irregular regular fractal patterns with sufficient accuracy. It give a new practical method for the determination of the fractal dimensions of ordinary fractal patterns and extension to researches on 2D fractal patterns and 2D regular fractal patterns may be envisaged. It gives the foundation for further researches on fractal patterns in nature.

Acknowledgement: The authors thank Prof. Duan Feng and Prof. Ming Wang for profitable discussions.

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