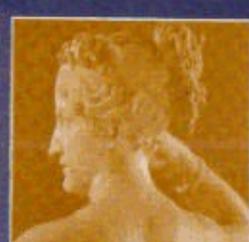
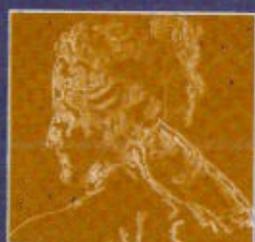
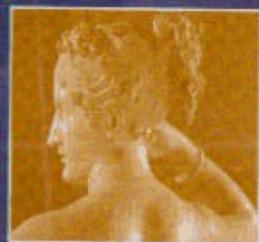


# IMAGE PROCESSING AND ANALYSIS

VARIATIONAL, PDE, WAVELET,  
AND STOCHASTIC METHODS



TONY F. CHAN  
JIANHONG (JACKIE) SHEN

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