

ALGEBRAIC NUMBER THEORY AND FERMAT'S LAST THEOREM

THIRD EDITION

$$x^n + y^n = z^n$$

$x^n + y^n \neq z^n$

$$y^n = z^n$$

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