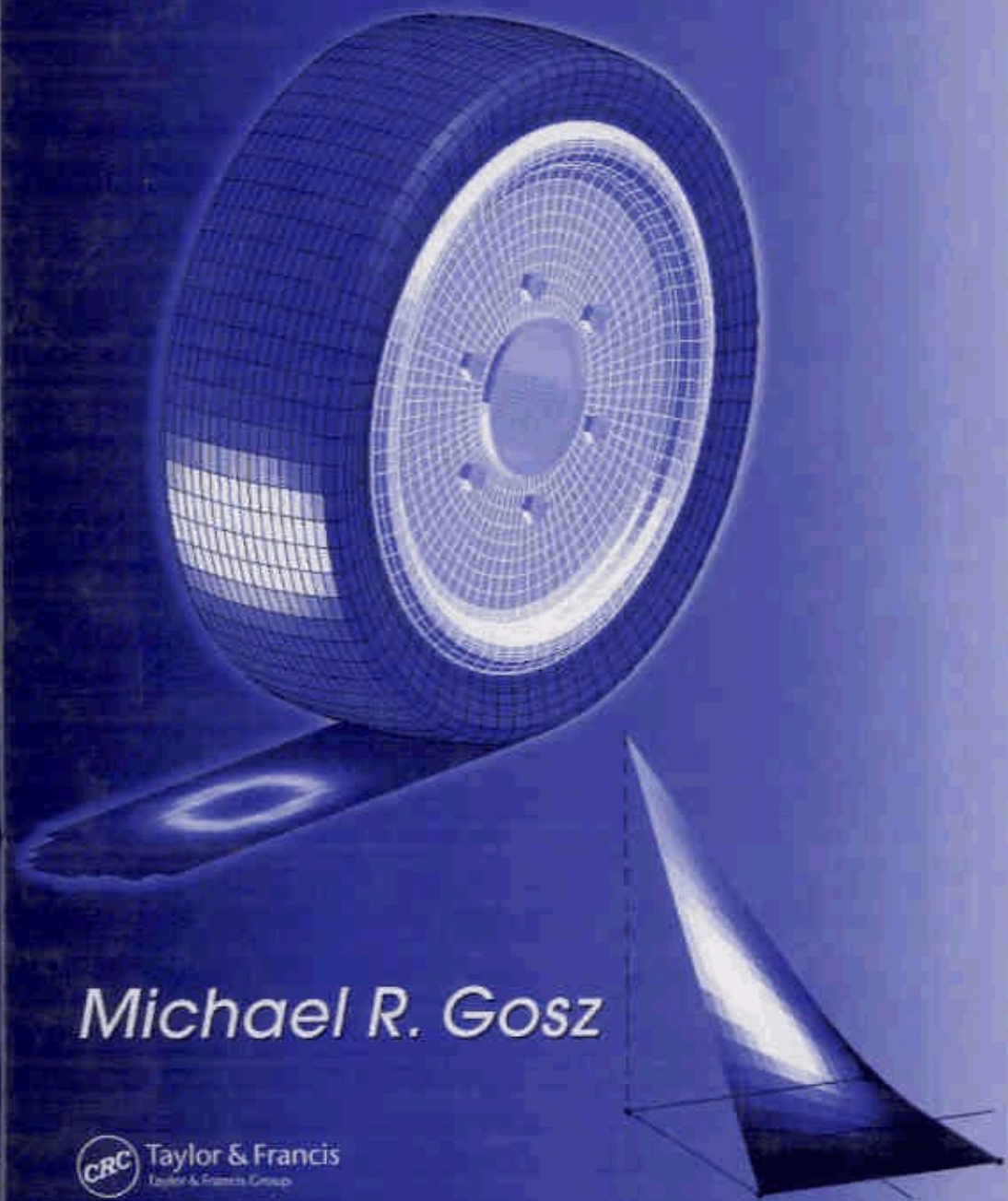
Finite Element Method

Applications in Solids, Structures, and Heat Transfer



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